Multiscale analysis of exit distributions for random walks in random environments

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Abstract

We present a multiscale analysis for the exit measures from large balls in $\mathbb{Z}^d$, $d \geq 3$, of random walks in certain i.i.d. random environments which are small perturbations of the fixed environment corresponding to simple random walk. Our main assumption is an isotropy assumption on the law of the environment, introduced by Bricmont and Kupianien. The analysis is based on propagating estimates on the variational distance between the exit measure and that of simple random walk, in addition to estimates on the variational distance between smoothed versions of these quantities.

1 Introduction

We consider random walks in random environments on $\mathbb{Z}^d$, $d \geq 3$, when the environment is a small perturbation of the fixed environment corresponding to simple random walk. More precisely, let $\mathcal{P}$ be the set of probability distributions on $\mathbb{Z}^d$, charging only neighbors of 0. If $\varepsilon \in (0, 1/2d)$, we set, with $\{e_i\}_{i=1}^d$ denoting the standard basis of $\mathbb{R}^d$,

$$\mathcal{P}_\varepsilon \stackrel{\mathrm{def}}{=} \left\{ q \in \mathcal{P} : \left| q\left(\pm e_i\right) - \frac{1}{2d} \right| \leq \varepsilon, \ \forall i \right\}. \tag{1.1}$$

$\Omega \stackrel{\mathrm{def}}{=} \mathcal{P}^{\mathbb{Z}^d}$ is equipped with the natural product σ-field $\mathcal{F}$. We call an element $\omega \in \Omega$ a *random environment*. For $\omega \in \Omega$, and $x \in \mathbb{Z}^d$, we consider the transition probabilities $p_\omega\left(x, y\right) \stackrel{\mathrm{def}}{=} \omega_x\left(y - x\right)$, if $|x - y| = 1$, and $p_\omega\left(x, y\right) = 0$ otherwise, and construct the random walk in random environment (RWRE) $\{X_n\}_{n \geq 0}$ with initial position $x \in \mathbb{Z}^d$ which is, given the environment ω, the Markov chain with $X_0 = x$ and transition probabilities

$$P_{\omega,x}(X_{n+1} = y | X_n = z) = \omega_z(y - z)\,.$$

(By a slight abuse of notation, for consistency with the sequel we also write $P_{\omega,x} = P_{p_\omega,x}$.)

We are mainly interested in the case of a *random* ω. Given a probability measure μ on $\mathcal{P}$, we consider the product measure $\mathbb{P}_\mu \stackrel{\mathrm{def}}{=} \mu^{\otimes \mathbb{Z}^d}$ on $\left(\Omega, \mathcal{F}\right)$. We usually drop the index μ in $\mathbb{P}_\mu$. In all that follows we make the following basic assumption

Condition 1.1
μ is invariant under lattice isometries, i.e. $\mu f^{-1} = \mu$ for any orthogonal mapping f which leaves $\mathbb{Z}^d$ invariant, and $\mu(\mathcal{P}_\varepsilon) = 1$ for some $\varepsilon \in (0, 1/2d)$ which will be specified later.

The model of RWRE has been studied extensively. We refer to [7] and [12] for recent surveys. A major open problem is the determination, for $d > 1$, of laws of large numbers and central limit theorems in full generality (the latter, both under the *quenched* measure, i.e. for $\mathbb{P}_\mu$-almost every ω, and under the *annealed* measure $\mathbb{P}_\mu \otimes P_{x,\omega}$). Although much progress has been reported in recent years ([1, 8, 9]), a full understanding of the model has not yet been achieved.

In view of the above state of affairs, attempts have been made to understand the perturbative behavior of the RWRE, that is the behavior of the RWRE when μ is supported on $\mathcal{P}_\varepsilon$ and ε is small. The first to consider such a perturbative regime were [2], who introduced Condition 1.1 and showed that in dimension $d \geq 3$, for small enough ε a quenched CLT holds[1]. Unfortunately, the multiscale proof in [2] is rather difficult, and challenging to follow. This in turns prompted the derivation, in [10], of an alternative multiscale approach, in the context of diffusions in random environments. One expects that the approach of [10] could apply to the discrete setup, as well.

Our goal in this paper is somewhat different: we focus on the exit law of the RWRE from large balls, and develop a multiscale analysis that allows us to conclude that the exit law approaches, in a suitable sense, the uniform measure. Like in [10], the hypothesis propagated involves smoothing. In [10], this was done using certain Hölder norms of (rescaled) transition probabilities. Here, we focus on two ingredients. The first is a propagation of the variational distance between the exit laws of the RWRE from balls and those of simple random walk (which distance remains small but does not decrease as the scale increases). The second is the propagation of the variation distance between the convolution of the exit law of the RWRE with the exit law of a simple random walk from a ball of (random) radius, and the corresponding convolution of the exit law of simple random walk with the same smoothing, which distance decreases to zero as scale increases (a precise statement can be found in Theorems 2.4 and 2.5; the latter, which is our main result, provides a local limit law for the exit measures). This approach is of a different nature than the one in [10] and, we believe, simpler. In future work we hope to combine our exit law approach with suitable exit time estimates in order to deduce a (quenched) CLT for the RWRE.

The structure of the article is the following. In the next section, we introduce our basic notation and state our induction step and our main results. In Section 3, we present our basic perturbation expansion, coarsening scheme for random walks, and auxiliary estimates for simple random walk. The (rather standard) proofs of the latter estimates are presented in the appendices. Section 4 is devoted to the propagation of the smoothed estimates, whereas Section 5 is devoted to the propagation of the variation distance estimate (the non-smooth estimate). Section 6 completes the proof of our main result by using the estimates of Sections 4 and 5.

[1] As the examples in [1] demonstrate, for every $d \geq 7$ and $\varepsilon > 0$ there are measures μ supported on $\mathcal{P}_\varepsilon$, with $\mathbb{E}_\mu \left[\sum_{i=1}^d e_i(q(e_i) - q(-e_i))\right] = 0$, such that $X_n/n \to_{n\to\infty} v \neq 0$, $\mathbb{P}_\mu$-a.s. One of the goals of Condition 1.1 is to prevent such situations from occurring.

2 Basic notation and main result

Sets: For $x \in \mathbb{R}^d$, $|x|$ is the Euclidean norm. If $A, B \subset \mathbb{Z}^d$, we set $d(A,B) \overset{\text{def}}{=} \inf\{|x-y| : x \in A, y \in B\}$. If $L > 0$, we write $V_L \overset{\text{def}}{=} \{x \in \mathbb{Z}^d : |x| \le L\}$, and for $x \in \mathbb{Z}^d$, $V_L(x) \overset{\text{def}}{=} x + V_L$. If $V \subset \mathbb{Z}^d$, $\partial V = \{x \in V^c : d(x,V) = 1\}$ is the outer boundary. If $x \in V$, we set $d_V(x) \overset{\text{def}}{=} d(x, \partial V)$. We also set $d_L(x) = L - |x|$ (note that $d_L(x) \neq d_{V_L}(x)$ with this convention). For $0 \le a < b \le L$, we define

$$\mathrm{Shell}_L(a,b) \overset{\text{def}}{=} \{x \in V_L : a \le d_L(x) < b\}, \ \mathrm{Shell}_L(b) \overset{\text{def}}{=} \mathrm{Shell}_L(0,b). \tag{2.1}$$

Functions: If F, G are functions $\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ we write FG for the (matrix) product: $FG(x,y) \overset{\text{def}}{=} \sum_u F(x,u)\, G(u,y)$, provided the right hand side is absolutely summable. F^k is the k-th power defined in this way, and $F^0(x,y) \overset{\text{def}}{=} \delta_{x,y}$. We interpret F also as a kernel, operating from the left on functions $f : \mathbb{Z}^d \to \mathbb{R}$, by $Ff(x) \overset{\text{def}}{=} \sum_y F(x,y)\, f(y)$. If $W \subset \mathbb{Z}^d$, we use 1_W not only as the indicator function but, by slight abuse of notation, also to denote the kernel $(x,y) \to 1_W(x)\, \delta_{x,y}$.

For a function $f : \mathbb{Z}^d \to \mathbb{R}$, $\|f\|_1 \overset{\text{def}}{=} \sum_x |f(x)|$, and $\|f\|_\infty \overset{\text{def}}{=} \sup_x |f(x)|$, as usual. If F is a kernel then, by an abuse of notation, we write $\|F\|_1$ for its norm as operator on L_∞, i.e.

$$\|F\|_1 \overset{\text{def}}{=} \sup_x \|F(x,\cdot)\|_1. \tag{2.2}$$

Transition probabilities: For transition probabilities $p = (p(x,y))_{x,y \in \mathbb{Z}^d}$, not necessarily nearest neighbor, we write $P_{p,x}$ for the law of a Markov chain $X_0 = x, X_1, \ldots$ having p as transition probabilities. If $V \subset \mathbb{Z}^d$, $\tau_V \overset{\text{def}}{=} \inf\{n \ge 0 : X_n \notin V\}$ is the first exit time from V, and $T_V \overset{\text{def}}{=} \tau_{V^c}$ the first entrance time. We set

$$\mathrm{ex}_V(x,z;p) \overset{\text{def}}{=} P_{p,x}(X_{\tau_V} = z).$$

For $x \in V^c$, one has $\mathrm{ex}_V(x,z;p) = \delta_{x,z}$. A special case is the standard simple random walk $p(x, \pm e_i) = 1/2d$, where $e_1, \ldots, e_d \in \mathbb{Z}^d$ is the standard base. We abbreviate this as p^{RW}, and set $P_x^{\mathrm{RW}} \overset{\text{def}}{=} P_{x,p^{\mathrm{RW}}}$. Also, exit distributions for the simple random walk are written as $\pi_V(x,z) \overset{\text{def}}{=} \mathrm{ex}_V\left(x,z;p^{\mathrm{RW}}\right)$.

We will coarse-grain *nearest-neighbor* transition probabilities p in the following way. Given $W \subset \mathbb{Z}^d$, we choose for any $x \in W$ either a fixed finite subset $U_x \subset W$, $x \in U_x$, or a probability distribution s_x on such sets. Of course, a fixed choice U_x is just a special choice for the distribution s_x, namely the one point distribution on U_x.

Definition 2.1
*A collection $\mathcal{S} = (s_x)_{x \in W}$ is called a **coarse graining scheme** on W. Given such a scheme, and nearest neighbor transition probabilities p, we define the coarse grained transitions by*

$$p^{\mathrm{CG}}_{\mathcal{S},W}(x,\cdot) \overset{\text{def}}{=} \sum_{U : x \in U \subset W} s_x(U)\, \mathrm{ex}_U(x,\cdot;p). \tag{2.3}$$

In the case of the standard nearest neighbor random walk, we use the notation $\pi_{\mathcal{S},W}$ instead of $\left(p^{\mathrm{RW}}\right)^{\mathrm{CG}}_{\mathcal{S},W}$.

Using the Markov property, we have, whenever W is finite,

$$\mathrm{ex}_W\left(x,\cdot;p\right)=\mathrm{ex}_W\left(x,\cdot;p^{\mathrm{CG}}_{\mathcal{S},W}\right). \tag{2.4}$$

We will choose the coarse-graining scheme in special ways. Fix once for all a probability density

$$\varphi:\mathbb{R}^+\to\mathbb{R}^+,\ \varphi\in C^\infty,\ \mathrm{support}\left(\varphi\right)=[1,2]\,. \tag{2.5}$$

If $m\in\mathbb{R}^+$, the rescaled density is defined by $\varphi_m\left(t\right)\overset{\mathrm{def}}{=}\left(1/m\right)\varphi\left(t/m\right)$. The image measure of $\varphi_m\left(t\right)dt$ under the mapping $t\to V_t\left(x\right)\cap W$ defines a probability distribution on subsets of W containing x. We may also choose m to depend on x, i.e. consider a field $\Psi=\left(m_x\right)_{x\in W}$ of positive real numbers on W. Such a field then defines via the above scheme coarse grained transition probabilities, which by a slight abuse of notation we denote as $p^{\mathrm{CG}}_{\Psi,W}$. In case $W=\mathbb{Z}^d$, we simply drop W in the notation. In case p is the standard nearest neighbor random walk, we write $\hat{\pi}_\Psi$ instead of p^{CG}_Ψ.

The random environment: We recall from the introduction the notation $\mathcal{P}_\varepsilon$, Ω, $p_\omega\left(x,y\right)$, and the natural product σ-field $\mathcal{F}$. For $A\subset\mathbb{Z}^d$, we write $\mathcal{F}_A=\sigma(\omega_x:x\in A)$. We also recall the probability measure μ on $\mathcal{P}$, the product measure $\mathbb{P}_\mu$, and Condition 1.1, which is assumed throughout.

For a random environment $\omega\in\Omega$, we typically write $\Pi_{V,\omega}\overset{\mathrm{def}}{=}\mathrm{ex}_V\left(\cdot,\cdot;p_\omega\right)$ and occasionally drop ω in the notation. So Π_V should always be understood as a *random* exit distribution. We will also use $\hat{\Pi}_{\mathcal{S},W}$ for $\left(p_\omega\right)^{\mathrm{CG}}_{\mathcal{S},W}$.

For $x\in\mathbb{Z}^d$, $L>0$, and $\Psi:\partial V_L\left(x\right)\to\mathbb{R}^+$, we define the random variables

$$D_{L,\Psi}\left(x\right)\overset{\mathrm{def}}{=}\left\|\left(\left[\Pi_{V_L(x)}-\pi_{V_L(x)}\right]\hat{\pi}_\Psi\right)\left(x,\cdot\right)\right\|_1, \tag{2.6}$$

$$D_{L,0}\left(x\right)\overset{\mathrm{def}}{=}\left\|\Pi_{V_L(x)}\left(x,\cdot\right)-\pi_{V_L(x)}\left(x,\cdot\right)\right\|_1, \tag{2.7}$$

and with $\delta>0$, we set

$$b_i\left(L,\Psi,\delta\right)\overset{\mathrm{def}}{=}\mathbb{P}\left(\left(\log L\right)^{-9+\frac{9(i-1)}{4}}<D_{L,\Psi}\left(0\right)\leq\left(\log L\right)^{-9+\frac{9i}{4}},\ D_{L,0}\left(0\right)\leq\delta\right),\ i=1,2,3,$$
$$b_4\left(L,\Psi,\delta\right)\overset{\mathrm{def}}{=}\mathbb{P}\left(\left\{\left(\log L\right)^{-2.25}<D_{L,\Psi}\left(0\right)\right\}\cup\left\{D_{L,0}\left(0\right)>\delta\right\}\right),$$
$$b\left(L,\Psi,\delta\right)\overset{\mathrm{def}}{=}\sum_{i=1}^4 b_i\left(L,\Psi,\delta\right).$$

We write $\mathcal{M}_L$ for the set of functions $\Psi:\partial V_L\to[L/2,2L]$ which are restrictions of functions defined on $\left\{x\in\mathbb{R}^d:L/2\leq|x|\leq 2L\right\}$ that have smooth third derivatives bounded by $10L^{-2}$ and fourth derivatives bounded by $10L^{-3}$. We write $\Psi_t=\left(m_x=t\right)_{x\in\mathbb{Z}^d}$ for the coarse-graining scheme that consists of constant coarse-graining at scale t. Of course, $\Psi_t\in\mathcal{M}_L$ for all t,L.

Condition 2.2
Let $L_1 \in \mathbb{N}$, and $\delta > 0$. We say that condition $\operatorname{Cond}(\delta, L_1)$ *holds provided that for all $L \leq L_1$, and for all $\Psi \in \mathcal{M}_L$,*

$$b_i(L, \Psi, \delta) \leq \frac{1}{4} \exp\left[-(1-(4-i)/13)(\log L)^2\right], \ i = 1,2,3,4. \tag{2.8}$$

In particular, if $\operatorname{Cond}(\delta, L_1)$ is satisfied, then for any $L \leq L_1$, and any $\Psi \in \mathcal{M}_L$,

$$\mathbb{P}\left(\{D_{L,0}(0) > \delta\} \cup \{D_{L,\Psi}(0) > (\log L)^{-9}\}\right) \leq \exp\left[-\frac{10}{13}(\log L)^2\right] \tag{2.9}$$

Our main technical inductive result is

Proposition 2.3
There exist $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$ there exists $\varepsilon_0(\delta)$ and $L_0 \in \mathbb{N}$ such that if $\varepsilon \leq \varepsilon_0$, $L_1 \geq L_0$, and μ is such that Condition 1.1 holds for ε, then

$$\operatorname{Cond}(\delta, L_1) \Longrightarrow \operatorname{Cond}\left(\delta, L_1(\log L_1)^2\right).$$

Given L_0, δ_0, we can always choose ε_0 so small that if Condition 1.1 is satisfied with $\varepsilon \leq \varepsilon_0$, then $\operatorname{Cond}(\delta_0, L_0)$ holds trivially. Proposition 2.3 then implies that for any $\delta < \delta_0$, there exists $\varepsilon_0 = \varepsilon_0(\delta)$ small enough such that if Condition 1.1 is satisfied with $\varepsilon \leq \varepsilon_0$, then $\operatorname{Cond}(\delta, L)$ holds for all L. In particular, one obtains immediately from Proposition 2.3 the following theorem (recall that Ψ_t denotes constant coarse-graining at scale t).

Theorem 2.4
For each $\delta > 0$ there exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that if Condition 1.1 is satisfied with $\varepsilon \leq \varepsilon_0$, then for any integer $r \geq 0$,

$$\limsup_{L\to\infty} L^r b(L, \Psi_L, \delta) = 0\,.$$

Our induction will also provide the following theorem which is the main result of our paper. It provides a local limit theorem for the exit law.

Theorem 2.5
There exists $\varepsilon_0 > 0$, such that if Condition 1.1 is satisfied with $\varepsilon \leq \varepsilon_0$, then for any $\delta > 0$, and for any integer $r \geq 0$,

$$\lim_{t\to\infty} \limsup_{L\to\infty} L^r b(L, \Psi_t, \delta) = 0\,.$$

The Borel-Cantelli lemma then implies that under the conditions of Theorem 2.4,

$$\limsup_{L\to\infty} D_{L,\Psi_t}(0) \leq c_t\,, \quad \mathbb{P}_\mu - a.s.,$$

where c_t is a constant such that $c_t \to_{t\to\infty} 0$.

A remark about the wording which we use. When we say that something holds for "large enough L", we mean that there exists L_0, *depending only on the dimension*, such that the statement holds for $L \geq L_0$. We emphasize that L_0 then *does not depend on ε.*

We write C for a generic positive constant, not necessarily the same at different occurrences. C may depend on the dimension d of the lattice, but on nothing else, except when indicated explicitly. Other constants, such as $c_0, c_1, \bar{c}, k_0, K, C_1$ etc., follow the same convention concerning what they depend on (d only, unless explicitly stated otherwise!), but their value is fixed throughout the paper and does not change from line to line.

3 Preliminaries

3.1 The perturbation expansion

Let $p = (p(x,y))_{x,y \in \mathbb{Z}^d}$ be a Markovian transition kernel on $\mathbb{Z}^d$, not necessarily nearest neighbor, but of finite range, and let $V \subset\subset \mathbb{Z}^d$. The Green kernel on V with respect to p is defined by

$$g_V(p)(x,y) \stackrel{\text{def}}{=} \sum_{k \geq 0} (1_V p)^k (x,y).$$

Evidently, if $z \notin V$, then

$$g_V(p)(\cdot, z) = \mathrm{ex}_V(\cdot, z; p). \tag{3.1}$$

If p, q are two transition kernels, write $\Delta_{p,q} = 1_V(p-q)$. The resolvent equation gives for every $n \in \mathbb{N}$,

$$\begin{aligned} & g_V(p) - g_V(q) = g_V(q)\Delta_{p,q} g_V(p) \\ & = \sum_{k=1}^{n-1} [g_V(q)\Delta_{p,q}]^k g_V(q) + [g_V(q)\Delta_{p,q}]^n g_V(p) = \sum_{k=1}^{\infty} [g_V(q)\Delta_{p,q}]^k g_V(q), \end{aligned} \tag{3.2}$$

assuming convergence of the infinite series, which will always be trivial in cases of interest to us, due to ellipticity and V being finite. We will occasionally slightly modify the above expansion, but the basis is always the first equality in (3.2).

3.2 The coarse graining schemes on V_L

Our proof of Theorems 2.4 and 2.5 is based on a couple of explicit coarse graining schemes, whose definitions we now present. Set

$$r(L) \stackrel{\text{def}}{=} L/(\log L)^{10}, \; s(L) \stackrel{\text{def}}{=} L/(\log L)^3, \; \mathrm{Sh}_L \stackrel{\text{def}}{=} \mathrm{Shell}_L(r(L)), \tag{3.3}$$

and

$$\gamma \stackrel{\text{def}}{=} \min\left(\frac{1}{10}, \frac{1}{2}\left(1 - \left(\frac{2}{3}\right)^{1/(d-1)}\right)\right). \tag{3.4}$$

We fix a C^∞-function $h : \mathbb{R}^+ \to \mathbb{R}^+$, which satisfies $h(u) = u$ for $u \leq 1/2$, $h(u) = 1$ for $u \geq 2$, and is strictly monotone and concave on $(1/2, 2)$. For $x \in V_L$, we set

$$h_L(x) \stackrel{\text{def}}{=} \gamma s(L) h\left(\frac{d_L(x)}{s(L)}\right). \tag{3.5}$$

Remark that for $d_L(x) \geq 2s(L)$, we have $h_L(x) = \gamma s(L)$.

Lemma 3.1
Fix $\delta_1 > 0$. Then, there is a constant $\bar{k}_0 = \bar{k}_0(\delta_1)$ such that if $k \geq \bar{k}_0(\delta_1)$, and $\Delta(x,y) = \Pi_{V_{kr(L)}(x)}(x,y) - \pi_{V_{kr(L)}(x)}(x,y)$ then for all L large, if for some $\delta > 0$, $d_L(x) \leq r(L)$ and $D_{kr(L),0}(x) \leq \delta$, then

$$\sum_{y \in V_L \cap \mathrm{Sh}_L} |\Delta(x,y)| \leq \delta + \delta_1 . \tag{3.6}$$

Proof. Fix k. We have

$$\begin{aligned}\sum_{y \in V_L \cap \mathrm{Sh}_L} |\Delta(x,y)| &\leq \Pi_{V_{kr(L)}(x)}(x, V_L \cap \mathrm{Sh}_L) + \pi_{V_{kr(L)}(x)}(x, V_L \cap \mathrm{Sh}_L) \\ &\leq \delta + 2\pi_{V_{kr(L)}(x)}(x, V_L \cap \mathrm{Sh}_L) .\end{aligned}$$

Choosing k large enough completes the proof. ■

We can now define our coarse graining schemes on V_L. The first will depend on a constant $k_0 > 1$ that will be chosen below, based on some a-priori estimates concerning simple random walk, see (3.20).

Definition 3.2

a) *The coarse graining scheme $\mathcal{S}_1 = \mathcal{S}_{1,L,k_0} = (s_x)_{x \in V_L}$ is defined for $d_L(x) \leq r(L)$ by $s_x = \delta_{V_{k_0 r(L)} \cap V_L}$, i.e. for such an x, the coarse graining is done by choosing the exit distribution from $V_{k_0 r(L)}(x) \cap V_L$. For $d_L(x) > r(L)$, we take $m_x = h_L(x)$ and define s_x according to the description following (2.5).*

b) *The coarse graining scheme $\mathcal{S}_2 = \mathcal{S}_{2,L} = (s_x)_{x \in V_L}$ is defined for all x by $m_x = h_L(x)$.*

We will need the second scheme only in Section 5, when propagating the part of the estimate $b_4(L, \Psi, \delta)$ involving the expression $D_{L,0}(x)$ of (2.7). Note that under $\mathcal{S}_2$, if $d_L(x) < 1/2\gamma$ then there is no coarse graining at all, i.e. $s_x = \delta_x$.

We write $\rho_{i,L}(x)$ for the range of the coarse graining scheme at x in scheme i, $i = 1, 2$, i.e.

$$\rho_{1,L}(x) \overset{\text{def}}{=} \begin{cases} k_0 r(L) & \text{for } d_L(x) \leq r(L) \\ 2h_L(x) & \text{for } r(L) < d_L(x) \end{cases} , \ \rho_{2,L} = 2h_L(x) . \tag{3.7}$$

3.3 Estimates on exit distributions and the Green's function

For notational convenience, we write π_L instead of π_{V_L}, and similarly in other expressions. For instance, we write τ_L instead of τ_{V_L}.

Lemma 3.3

a) *For $x \in \partial V_L$,*

$$\frac{1}{C} L^{-d+1} \leq \pi_L(x) \leq C L^{-d+1}.$$

b) *Let x be a vector of unit length in $\mathbb{R}^d$, let $0 < \theta < 1$, and define the cone $C_\theta(x) \overset{\text{def}}{=} \left\{ y \in \mathbb{Z}^d : \langle y, x \rangle \geq (1-\theta)|y| \right\}$. For any θ, there exists $\eta(\theta) > 0$, such that for all L large enough, and all x*

$$\pi_L(0, C_\theta(x)) \geq \eta(\theta) . \tag{3.8}$$

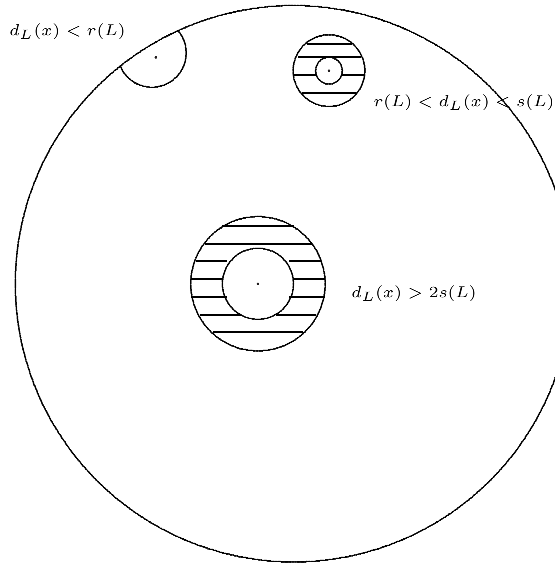

Figure 1: The coarse graining scheme $\mathcal{S}_1$

c) Let $0 < l < L$, and $x \in \mathbb{Z}^d$ satisfy $l < |x| < L$. Then

$$P_x^{\mathrm{RW}}\left(\tau_L < T_{V_l}\right) = \frac{l^{-d+2} - |x|^{-d+2} + O\left(l^{-d+1}\right)}{l^{-d+2} - L^{-d+2}}$$

Proof. a) is Lemma 1.7.4 of [5]. b) is immediate from a). c) is Proposition 1.5.10 of [5]. ■

We will repeatedly make use of the following lemma.

Lemma 3.4
Assume $x, y \in V_L$, $1 \leq a \leq 5d_L(y)$, $x \notin V_{2a}(y)$. Then

$$P_x\left(T_{V_a(y)} < \tau_{V_L}\right) \leq C\frac{a^{d-2} d_L(y)\, d_L(x)}{|x-y|^d} \tag{3.9}$$

The proof will be given in Appendix A.

We will need a corresponding result for the Brownian motion. We write $\pi_L^{\mathrm{BM}}(y, dy')$ for the exit distribution of the Brownian motion from the ball C_L of radius L in $\mathbb{R}^d$. The following lemma is an easy consequence of the Poisson formula, see [5, (1.43)].

Lemma 3.5
For any $y \in C_L$, it holds that

$$\frac{C^{-1} d(y, \partial C_L)}{|y-y'|^d} \leq \frac{\pi_L^{\mathrm{BM}}(y, dy')}{dy'} \leq \frac{C d(y, \partial C_L)}{|y-y'|^d}, \tag{3.10}$$

where dy' is the surface measure on ∂C_L.

We will also need a comparison between smoothed exit distribution of the random walk, and that of Brownian motion. Given $L > 0$, and $\Psi \in \mathcal{M}_L$, let

$$\phi_{L,\Psi} \stackrel{\mathrm{def}}{=} \pi_L \hat{\pi}_\Psi. \tag{3.11}$$

We consider also the corresponding Brownian kernel on $\mathbb{R}^d$,

$$\phi_{L,\Psi}^{\mathrm{BM}}(y,dz) \stackrel{\mathrm{def}}{=} \int_{\partial C_L(0)} \pi_{C_L(0)}^{\mathrm{BM}}(y,dw) \int \pi_{C_t(w)}^{\mathrm{BM}}(w,dz)\, \varphi_{m_w}(t)\, dt, \tag{3.12}$$

where $\Psi = (m_w)$, and where we write $\phi_{L,\Psi}^{\mathrm{BM}}(y,z)$ for the density of $\phi_{L,\Psi}^{\mathrm{BM}}(y,dz)$ with respect to d-dimensional Lebesgue measure.

Lemma 3.6
There exists a constant C such that for $L > 0$, and $\Psi \in \mathcal{M}_L$, we have

$$\sup_{y \in V_L} \sup_{z \in \mathbb{Z}^d} \left| \phi_{L,\Psi}(y,z) - \phi_{L,\Psi}^{\mathrm{BM}}(y,z) \right| \leq C L^{-d-1/5}.$$

Lemma 3.7
There exists a constant C such that for $L > 0$ and $\Psi \in \mathcal{M}_L$, we have

$$\sup_{y,z} \left\| \partial_y^i \phi_{L,\Psi}^{\mathrm{BM}}(y,z) \right\| \leq C L^{-d-i}, \, i = 1,2,3.$$

The proofs of these two lemmas are again in Appendix A.

We can draw two immediate conclusions from these results:

Proposition 3.8

a) *Let y, y' be in V_L, and $\Psi \in \mathcal{M}_L$. Then*

$$\left| \phi_{L,\Psi}(y,z) - \phi_{L,\Psi}(y',z) \right| \leq C \left(L^{-d-1/5} + |y - y'| \, L^{-d-1} \right). \tag{3.13}$$

b) *Let $x \in V_L$, and l be such that $V_l(x) \subset V_L$. Consider a signed measure μ on V_l with total mass 0 and total variation norm $|\mu|$, which is invariant under lattice isometries. Then*

$$\left| \sum\nolimits_y \mu(y-x)\, \phi_{L,\Psi}(y,z) \right| \leq C |\mu| \left(L^{-d-1/5} + \left(\frac{l}{L} \right)^3 L^{-d} \right). \tag{3.14}$$

Proof of Proposition 3.8. a) is immediate from Lemmas 3.6 and 3.7. As for b), we get from Lemma 3.6 that

$$\left| \sum\nolimits_y \mu(y-x)\, \phi_{L,\Psi}(y,z) - \sum\nolimits_y \mu(y-x)\, \phi_{L,\Psi}^{\mathrm{BM}}(y,z) \right| \leq C |\mu| \, L^{-d-1/5},$$

while

$$\begin{aligned} \sum\nolimits_y \mu(y-x)\, \phi_{L,\Psi}^{\mathrm{BM}}(y,z) &= \sum\nolimits_y \mu(y-x) \left[\phi_{L,\Psi}^{\mathrm{BM}}(y,z) - \phi_{L,\Psi}^{\mathrm{BM}}(x,z) \right] \\ &= \sum\nolimits_y \mu(y-x)\, \partial_x \phi_{L,\Psi}^{\mathrm{BM}}(x,z) [y-x] \\ &\quad + \frac{1}{2} \sum\nolimits_y \mu(y-x)\, \partial_x^2 \phi_{L,\Psi}^{\mathrm{BM}}(x,z) [y-x, y-x] + R(\mu,x,z), \end{aligned} \tag{3.15}$$

where, due to Lemma 3.7,

$$|R(\mu,x,z)| \leq C |\mu| \left(\frac{l}{L} \right)^3 L^{-d} \tag{3.16}$$

uniformly in x and z, and $\partial^k F[u_1, \ldots, u_k]$ denotes the k-th derivative of a function F in directions $u_1, \ldots, u_k$. The first summand on the right hand side of (3.15) vanishes because μ has mean 0. The second vanishes because by the invariance under lattice isometry of μ, the summand involves only the Laplacian of $\phi^{\mathrm{BM}}_{L,\Psi}(\cdot, z)$, which in turn vanishes because of harmonicity of $\pi^{\mathrm{BM}}_{C_L(0)}(x, \cdot)$ in the x-variable. The proof of the proposition is complete. ■

The next lemma gives a-priori estimates for coarse-grained walks. We use $\hat{\pi}^{(i)}_L$, $i = 1, 2$, to denote the transitions of the coarse grained random walk that uses the coarse graining $\mathcal{S}_i$, and $\hat{g}^{(i)}_L$ to denote the corresponding Green's function. Note that these quantities all depend on L and k_0, but we suppress these from the notation. Recall that $\mathrm{Sh}_L = \mathrm{Shell}_L(r(L))$, c.f. (3.3).

Lemma 3.9
There exists a constant C (independent of k_0!) such that:

a)
$$\sup_{x \in V_L} \hat{g}^{(1)}_L(x, \mathrm{Sh}_L) \le C.$$

b) If $i = 1$ and $r(L) \le a \le 3s(L)$ or $i = 2$ and $a \le 3s(L)$ then,
$$\sup_{x \in V_L} \hat{g}^{(i)}_L(x, \mathrm{Shell}_L(a, 2a)) \le C.$$

c) For all $x, y \in V_L \setminus \mathrm{Shell}_L(s(L))$, and $i = 1, 2$,
$$\hat{g}^{(i)}_L(x, y) \le C \begin{cases} \frac{1}{s(L)^2 [|x-y| \vee s(L)]^{d-2}}, & y \neq x \\ 1, & y = x\,. \end{cases}$$

d) For $i = 1, 2$,
$$\sup_{x \in V_L} \hat{g}^{(i)}_L(x, V_L) \le C(\log L)^6\,.$$

e) For $i = 1, 2$,
$$\sup_{x, x' \in V_L : |x - x'| \le s(L)} \sum_{y \in V_L} \left| \hat{g}^{(i)}_L(x, y) - \hat{g}^{(i)}_L(x', y) \right| \le C(\log L)^3$$

The proof is presented in Appendix B.

Lemma 3.9 plays a crucial role in our smoothing procedure. As a preparation, for $k \ge 1$, set
$$B_1(k) \stackrel{\mathrm{def}}{=} \mathrm{Shell}_L\left((4/3)^k r(L)\right). \tag{3.17}$$

$B_1(k) \subset \mathrm{Shell}_L(s(L))$ if $k \le 20 \log\log L$. By Lemma 3.9, there exists a constant $\bar{c} \ge 1$ (again, independent of k_0!) such that

$$\sup_{x \in V_L} \hat{g}^{(1)}_L(x, B_1(k)) \le \bar{c} \begin{cases} k\,, & \text{if } k \le 20 \log\log L \\ (\log L)^6 & \text{if } k > 20 \log\log L \end{cases}. \tag{3.18}$$

and, for any ball $V_{rs(L)}(z) \subset V_{L - s(L)}$, $r \ge 1$,
$$\sup_{x \in V_L} \hat{g}^{(1)}_L\left(x, V_{rs(L)}(z)\right) \le \bar{c} r^d\,. \tag{3.19}$$

With $\bar{c}$ as in (3.18) and (3.19), we fix the constant k_0 large enough such that:

$$\begin{aligned} k_0 &\geq \bar{k}_0(1/200\bar{c}), \\ \sup_{x\in \mathrm{Sh}_L} P_x^{\mathrm{RW}}\left(\tau_{V_L} < \tau_{V_{k_0 r(L)}(x)}\right) &\geq 9/10, \\ \sup_{x\in \mathrm{Sh}_L} \pi_{V_{k_0 r(L)}(x)}\left(x, V_L\right) &\leq 17/32. \end{aligned} \tag{3.20}$$

That the two last estimates in (3.20) hold for k_0 large is obvious, for example from Donsker's invariance principle.

4 Smoothed exits

In this section, we provide estimates on the quantity $D_{L,\Psi}(0)$. We use the perturbation expansion in (3.2) repeatedly. The main application is in comparing exit distributions, as follows. If $V \subset\subset \mathbb{Z}^d$, and $\mathcal{S}$ is any coarse graining scheme on V (as in Definition 2.1), we compare the exit distribution of the RWRE Π_V with the exit distribution π_V of simple random walk through this perturbation expansion, using however coarse grained transitions inside V : using (3.1) and (2.4) we get for $x \in V$

$$\left(\Pi_V - \pi_V\right)(x, \cdot) = \sum_{k=0}^{\infty} \left(\hat{g}_{\mathcal{S},V}\left[\Delta_{\mathcal{S},V}\hat{g}_{\mathcal{S},V}\right]^k \Delta_{\mathcal{S},V}\pi_V\right)(x, \cdot),$$

where

$$\Delta_{\mathcal{S},V} \overset{\mathrm{def}}{=} 1_V\left(\hat{\Pi}_{\mathcal{S},V} - \hat{\pi}_{\mathcal{S},V}\right), \ \hat{g}_{\mathcal{S},V} \overset{\mathrm{def}}{=} g_V\left(\hat{\pi}_{\mathcal{S},V}\right).$$

Throughout this section, we consider only the coarse graining scheme $\mathcal{S} = \mathcal{S}_1$ as in Definition 3.2. We keep L and V_L fixed, and drop throughout the $\mathcal{S}, V$ subscripts, writing $\hat{\Pi}, \hat{\pi}, \Delta$ and $\hat{g}$ for $\hat{\Pi}_{\mathcal{S},V}, \hat{\pi}_{\mathcal{S},V}, \Delta_{\mathcal{S},V}$ and $\hat{g}_{\mathcal{S},V}$. We use repeatedly the identity

$$\hat{g}(x, \cdot) = \delta_{x,\cdot} + \hat{\pi}\hat{g}(x, \cdot), \ x \in V_L.$$

Setting, for $k \geq 1$,

$$\zeta^{(k)} = \Delta^{k-1}\left(\Delta\hat{\pi}\hat{g}\right), \tag{4.1}$$

we get

$$\Pi_L - \pi_L = \hat{g}\sum_{m=1}^{\infty}\sum_{k_1,\ldots,k_m=1}^{\infty} \zeta^{(k_1)} \cdot \ldots \cdot \zeta^{(k_{m-1})}\Delta^{k_m}\pi_L \overset{\mathrm{def}}{=} \mathcal{R}_L. \tag{4.2}$$

Remark that we can replace in $\zeta^{(k)}$ the second part:

$$\left(\Delta\hat{\pi}\hat{g}\right)(x, y) = \sum_z \left(\Delta\hat{\pi}\right)(x, z)\left(\hat{g}(z, y) - \hat{g}(x, y)\right),$$

i.e., we gain a discrete derivative in the Green function.

We can now describe informally our basic strategy. When analyzing the term $D_{L,\Psi}(0)$, boundary effect are not essential, and one can consider all steps to be coarse-grained (some extra care is however needed near the boundary, which leads to the specific form of the coarse graining scheme $\mathcal{S}_1$, but we gloss over these details in the description that follows). Note that the steps of the coarse-grained random walk are essentially in the scale $L/(\log L)^3$. In this scale, most $x \in V_L$ are good, that

is the individual steps of the coarse-grained random walk are controlled by the good event in the induction hypothesis. Consider the linear term in (4.2), that is the term with $m = 1$, which turns out to be the dominant term in the expansion. Suppose first all $x \in V_L$ are good, and consider the term with $k_1 = 0$. In this case, each term is smoothed at scale L from the right, and its variational norm is bounded by $o((\log L)^{-3})O((\log L)^{-9})$. A-priori estimates on the coarse-grained simple random walk yield that the sum over the coarse grained Green function $\hat{g}$ is $O((\log L)^{-6})$. This would look alarming, as multiplying these gives rise to an error which is only $o((\log L)^{-6})$, which could result in non-propagation of the induction hypothesis. However, one can use the fact that the individual contributions from sites distance by $\rho_{1,L}$ are independent, and of zero mean due to the isotropy assumption. Averaging over this sum of essentially independent random variables improves the estimate from the worst-case value of $o((\log L)^{-6})$ back to the desired value of $o((\log L)^{-9})$, see the proof of Proposition 4.3. The terms with $k_1 \geq 1$ are handled similarly, using now the part of the induction hypothesis involving $D_{L,0}(0)$ to control the extra powers of Δ and ensure the convergence of the series. A similar strategy is applied to the "non-linear" terms with $m > 1$. Boundary terms are handled by using the fact that the coarse grained random walk is unlikely to stay at distance less than $r(L)$ from the boundary for many steps.

A major complication in handling the perturbation expansion is the presence of "bad regions". The advantage of the coarse graining scheme $\mathcal{S} = \mathcal{S}_1$ is that it is unlikely to have more than one "bad region", and that this single bad region can be handled by an appropriate surgery, once appropriate Green function estimates for the RWRE in a "good environment" are derived, see Section 4.3.

We now turn to the actual proof, and write $B_L^{(i)}$, $i = 1, 2, 3, 4$, for the collection of points which are bad on level i, and in the right scale, with respect to the coarse graining scheme $\mathcal{S}_1$. That is, for $i = 1, 2, 3$,

$$\begin{aligned} B_L^{(i)} =& \{x \notin \mathrm{Sh}_L : D_{r,h_L(x)}(x) > (\log L)^{-9+\frac{9(i-1)}{4}} \text{ for some } \ r \in [h_L(x), 2h_L(x)], \\ & D_{r,h_L(x)}(x) \leq (\log L)^{-9+\frac{9i}{4}} \text{ for all } r \in [h_L(x), 2h_L(x)]\,,\ D_{r,0}(x) \leq \delta\}\,, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} B_L^{(4)} =& \{x \notin \mathrm{Sh}_L : D_{r,h_L(x)}(x) > (\log L)^{-\frac{9}{4}} \text{ or } D_{r,0}(x) > \delta\,, \\ & \text{for some } \ r \in [h_L(x), 2h_L(x)]\} \bigcap \{x \in \mathrm{Sh}_L : D_{k_0 r(L),0}(x) \geq \delta\}\,. \end{aligned} \tag{4.4}$$

We also write

$$B_L \stackrel{\text{def}}{=} \bigcup_{i=1}^{4} B_L^{(i)}\,,\ \mathrm{Good}_L \stackrel{\text{def}}{=} \{B_L = \emptyset\}\,. \tag{4.5}$$

As mentioned in the beginning of this section, a major complication in handling the perturbation expansion is the "bad regions". The advantage of the coarse graining scheme $\mathcal{S} = \mathcal{S}_1$ is that it is unlikely to have essentially more than one "bad region". To make this statement precise, note that if $L_1 \leq L \leq L_1 (\log L_1)^2$ then all the radii involved in the definition of badness are smaller than L_1, if L_1 is chosen large enough. Remark also that if $d_L(x) > r(L)$, then $h_L(x + \cdot) \in \mathcal{M}_r$ for $h_L(x) \leq r \leq 2h_L(x)$, and therefore, if L_1 is large enough, $\mathrm{Cond}(\delta, L_1)$ holds, and

$L_1 \leq L \leq L_1 (\log L_1)^2$, then

$$\mathbb{P}(x \in B_L) \leq 2\gamma s(L) \exp\left[-\frac{10}{13}\left(\log \frac{\gamma L}{(\log L)^{10}}\right)^2\right] \leq \exp\left[-0.7 (\log L)^2\right]. \quad (4.6)$$

The points y whose random environment ω_y can influence the badness of x are evidently within radius $\rho_L(x) = \rho_{1,L}(x)$ from x, see (3.7). If $|x-y| > \rho_L(x) + \rho_L(y)$, then $\{x \in B_L\}$ and $\{y \in B_L\}$ are independent. Therefore, if we define

$$\mathrm{TwoBad}_L \stackrel{\mathrm{def}}{=} \bigcup_{x,y \in V_L : |x-y| > \rho_L(x)+\rho_L(y)} \{x \in B_L\} \cap \{y \in B_L\}, \quad (4.7)$$

then:

Lemma 4.1
Assume L_1 large enough, (2.8) for L_1, and $L_1 \leq L \leq L_1 (\log L_1)^2$. Then

$$\mathbb{P}(\mathrm{TwoBad}_L) \leq \exp\left[-1.2 (\log L)^2\right].$$

Next, we regard $\hat{\Pi}$ as a field $\left(\hat{\Pi}(x,\cdot)\right)_{x \in V_L}$ of random transition probabilities. We defined the "goodified" transition probabilities

$$\mathrm{gd}\left(\hat{\Pi}\right)(x,\cdot) \stackrel{\mathrm{def}}{=} \begin{cases} \hat{\Pi}(x,\cdot) & \text{if } x \notin B_L \\ \hat{\pi}(x,\cdot) & \text{if } x \in B_L \end{cases}. \quad (4.8)$$

This field might no longer come from an i.i.d. RWRE, but nevertheless, we have the property that $\mathrm{gd}\left(\hat{\Pi}_L\right)(x,\cdot)$ and $\mathrm{gd}\left(\hat{\Pi}_L\right)(y,\cdot)$ are independent provided $|x-y| > \rho_L(x) + \rho_L(y)$. If X is a random variable depending on ω only trough $\hat{\Pi}_L$ we define $\mathrm{gd}(X)$ by replacing $\hat{\Pi}_L$ by $\mathrm{gd}\left(\hat{\Pi}_L\right)$.

We next take $\Psi \in \mathcal{M}_L$, and set $\phi \stackrel{\mathrm{def}}{=} \phi_{L,\Psi}$, as in (3.11). An easy consequence of our definitions and Lemma 3.1 is the following.

Lemma 4.2
If $\delta \leq (1/800\bar{c})$ then, for all $x \in V_L$ and $k \geq 2$,

$$\mathbf{1}_{\{B_L=\emptyset\}} \|\Delta^k(x,\cdot)\|_1 \leq \frac{1}{\bar{c}}\left(\frac{1}{8}\right)^k. \quad (4.9)$$

Proof. Since $\max_{x \in V_L} \|\Delta(x,\cdot)\|_1 \leq 2$ and $\bar{c} \geq 1$, it is enough to prove that

$$\mathbf{1}_{\{B_L=\emptyset\}} \sum_{z \in V_L} |\Delta^2(x,z)| \leq \left(\frac{1}{64\bar{c}}\right).$$

If $x \notin \mathrm{Sh}_L$ then, on the event $\{B_L = \emptyset\}$, $\|\Delta(x,\cdot)\|_1 \leq \delta$ and hence $\|\Delta^2(x,\cdot)\|_1 \leq 2\delta \leq 1/64\bar{c}$ due to our choice of δ. On the other hand, if $x \in \mathrm{Sh}_L$ then on the event

$\{B_L = \emptyset\}$,

$$\sum_{z\in V_L} |\Delta^2(x,z)| = \sum_{z\in V_L} \left|\sum\nolimits_{y\in V_L} \Delta(x,y)\Delta(y,z)\right| \tag{4.10}$$
$$\le 2\left|\sum_{y\in \mathrm{Sh}_L} \Delta(x,y)\right| + \left|\sum_{y\in V_L\setminus \mathrm{Sh}_L} \Delta(x,y)\right| \max_{y\in V_L\setminus \mathrm{Sh}_L} \sum_{z\in V_L} |\Delta(y,z)|$$
$$\le 2(\delta + \frac{1}{200\bar{c}}) + 2\delta = 4\delta + \frac{1}{100\bar{c}} < \frac{1}{64\bar{c}}\,,$$

where Lemma 3.1 and $k_0 \ge \bar{k}_0(1/200\bar{c})$ were used in the next to last inequality. ■

In what follows, we will always consider $\delta \le 1/800\bar{c}$.

4.1 The linear part

For $x \in V_L,\ B \subset V_L$, set

$$\xi_x^{(k)}(B,z) = \sum_{y\in B} \hat{g}(x,y)\left(\Delta^k \pi_L \hat{\pi}_\Psi\right)(y,z) \tag{4.11}$$
$$= \sum_{y\in B}\sum_{y'\in V_L} \hat{g}(x,y)\,\Delta^k(y,y')\left(\phi(y',z) - \phi(y,z)\right)\,,$$

where the last equality is because the total mass of $\Delta(y,\cdot)$ vanishes.

We write $\xi_x^{(k)}(z)$ for $\xi_x^{(k)}(V_L,z)$; in the notation of (4.2), $\xi_x^{(k)}(z) = \zeta^{(k)}\hat{\pi}_\Psi(x,z)$. Define

$$G_L \overset{\text{def}}{=} \left\{\sup_{x\in V_L} \sum\nolimits_{k\ge 1} \left\|\xi_x^{(k)}\right\|_1 \le (\log L)^{-37/4}\right\}.$$

G_L is precisely the event that the $m = 1$ term in the perturbation expansion (4.2), smoothed by $\hat{\pi}_\Psi$, is "small".

Proposition 4.3
If L is large enough, then

$$\mathbb{P}\left((G_L)^c \cap \mathrm{Good}_L\right) \le \exp\left[-(\log L)^{17/8}\right].$$

Proof. It suffices to prove that

$$\sup_{x\in V_L} \mathbb{P}\left(\sum\nolimits_{k\ge 1}\left\|\xi_x^{(k)}\right\|_1 \ge (\log L)^{-37/4},\ \mathrm{Good}_L\right) \le \exp\left[-(\log L)^{9/4}\right].$$

Note that

$$\mathbb{P}\left(\sum\nolimits_{k\ge 1}\left\|\xi_x^{(k)}\right\|_1 \ge (\log L)^{-37/4},\ \mathrm{Good}_L\right)$$
$$= \mathbb{P}\left(\sum\nolimits_{k\ge 1}\left\|\mathrm{gd}\left(\xi_x^{(k)}\right)\right\|_1 \ge (\log L)^{-37/4},\ \mathrm{Good}_L\right)$$
$$\le \mathbb{P}\left(\sum\nolimits_{k\ge 1}\left\|\mathrm{gd}\left(\xi_x^{(k)}\right)\right\|_1 \ge (\log L)^{-37/4}\right).$$

For notation convenience, we drop the notation $\mathrm{gd}(\cdot)$, and just use the fact that all $\hat{\Pi}$ involved satisfy the appropriate "goodness" properties. (Remark that after "goodifications", the distribution of $\hat{\Pi}(x, x+\cdot)$ remains invariant under lattice isometries, provided $d_L(x) > 2s(L)$.)

We split $\xi_x^{(k)}$ into different parts. If $y \notin \mathrm{Sh}_L$ and $\Delta(y,y') > 0$, we have, since $\gamma \leq 1/8$, that $|y-y'| \leq d_L(y)/4$, i.e. $d_L(y) \leq (4/3)\, d_L(y')$. Therefore, if $y' \in \mathrm{Sh}_L$ and $\Delta^k(y,y') > 0$, then $d_L(y) \leq (4/3)^k r(L)$. Recall the set $B_1(k)$, c.f. (3.17), and the estimate (3.18). If $y \in B_1(k)$, and $\Delta^k(y,y') > 0$, we have

$$|y-y'| \leq kk_0 r(L) + 3^k \max(r(L), d_L(y)) \leq \left(kk_0 + 4^k\right) r(L),$$

and applying (3.13), we see that for $y \in B_1(k)$, and y' such that $\Delta^k(y,y') > 0$, we have

$$|\phi(y,z) - \phi(y',z)| \leq C\left(kk_0 + 4^k\right) L^{-d} (\log L)^{-10}.$$

By Lemma 4.2, we have $\left\|\Delta^k(y,\cdot)\right\|_1 \leq 8^{-k}$. Combining all these estimates with parts b) and d) of Lemma 3.9, we have

$$\left\|\xi_x^{(k)}(B_1(k))\right\|_1 \leq C \begin{cases} 8^{-k}\left(kk_0 + 4^k\right)(\log L)^{-10}, & \text{if } k \leq 20 \log\log L, \\ 8^{-k}\left(kk_0 + 4^k\right)(\log L)^{-4}, & \text{if } k > 20 \log\log L. \end{cases} \tag{4.12}$$

(We emphasize our convention regarding constants, and in particular the fact that C does not depend on x.) Hence,

$$\sup_x \sum_{k \geq 1} \left\|\xi_x^{(k)}(B_1(k))\right\|_1 \leq C(\log L)^{-10} \leq (\log L)^{-37/4}/3. \tag{4.13}$$

Next, let

$$B_2(k) \stackrel{\mathrm{def}}{=} \mathrm{Shell}_L\left((4/3)^k r(L), (5/4)^k 2s(L)\right).$$

If $y \in B_2(k)$ and $\Delta^k(y,y') > 0$, we have $d_L(y') > r(L)$, and we get, using the fact that for $x \notin \mathrm{Sh}_L$ one can write $\pi_L(x,\cdot) = (\hat{\pi}\pi_L)(x,\cdot)$,

$$\xi_x^{(k)}(B_2(k),z) = \sum_{y \in B_2(k)} \sum_{y' \in V_L} \hat{g}(x,y)\, D_k(y,y')\,(\phi(y,z) - \phi(y',z)),$$

where

$$D_k \stackrel{\mathrm{def}}{=} \Delta^k \hat{\pi}, \tag{4.14}$$

and, on a "good" environment,

$$\begin{aligned} \sup_{y \in B_2(k)} \|D_k(y,\cdot)\|_1 &\leq \sup_{y \in B_2(k)} \left\|\Delta^{k-1}(y,\cdot)\right\|_1 \sup_{x: d_L(x) > r(L)} \|\Delta\hat{\pi}(x,\cdot)\|_1 \\ &\leq C 8^{-k} (\log L)^{-9}. \end{aligned} \tag{4.15}$$

Using Lemma 3.9 b), we have $\sup_x \hat{g}(x, \mathrm{Shell}_L(3s(L))) \leq C \log\log L$. Put

$$A_j \stackrel{\mathrm{def}}{=} \mathrm{Shell}_L\left((2 + (j-1)/4)\, s(L), (2 + j/4)\, s(L)\right), j \geq 1.$$

Starting from a point in A_j, $j \geq 3$, the coarse grained simple random walk has a probability $\geq 1/C$ to reach A_{j-2} in one step. Starting from A_{j-2}, an ordinary random walk has a probability $\geq 1/C$ to leave $V_{L+k_0 r(L)}$ before reaching A_j, and therefore, the coarse grained simple random walk leaves V_L before reaching A_j with at least the same probability. Therefore $\sup_x \hat{g}(x, A_j) \leq Cj$, and thus,

$$\sup_x \hat{g}(x, B_2(k)) \leq C\left(\left(\frac{5}{4}\right)^{2k} + \log\log L\right) \leq C\left(2^k + \log\log L\right).$$

If $y \in B_2(k)$, and $\Delta^k(y, y') > 0$, then $|y - y'| \leq 2ks(L)$, and therefore,

$$|\phi(y, z) - \phi(y', z)| \leq CkL^{-d} (\log L)^{-3},$$

again by (3.13). Therefore, we get,

$$\begin{aligned}\left\| \xi_x^{(k)} (B_2(k), \cdot) \right\|_1 &\leq Ck (\log L)^{-12} \left[4^{-k} + 8^{-k} \log\log L \right], \\ \sup_x \sum_{k \geq 1} \left\| \xi_x^{(k)} (B_2(k), \cdot) \right\|_1 &\leq (\log L)^{-37/4} /3. \end{aligned} \tag{4.16}$$

Let $B_3(k) \stackrel{\text{def}}{=} V_L \setminus (B_1(k) \cup B_2(k))$. Given $j \in \mathbb{Z}$, let

$$I_j \stackrel{\text{def}}{=} \{jks(L) + 1, \ldots, (j+1) ks(L)\}.$$

Then for $\mathbf{j} \in \mathbb{Z}^d$, put $W_{\mathbf{j},k} \stackrel{\text{def}}{=} B_3(k) \cap I_{j_1} \times \cdots \times I_{j_d}$, with diameter $(W_{\mathbf{j}}) \leq \sqrt{d} ks(L)$. Let J_k be the set of $\mathbf{j}$'s for which these sets are not empty. We subdivide J_k into subsets $J_{1,k}, \ldots, J_{K(d,k),k}$ such that for any $1 \leq \ell \leq K(d,k)$,

$$\mathbf{j}, \mathbf{j}' \in J_{\ell,k},\ \mathbf{j} \neq \mathbf{j}' \Longrightarrow d(W_{\mathbf{j},k}, W_{\mathbf{j}',k}) > ks(L). \tag{4.17}$$

We set, recalling (4.14),

$$\xi_{x,\mathbf{j}}^{(k)}(z) \stackrel{\text{def}}{=} \sum_{y \in W_{\mathbf{j},k}} \sum_{y' \in V_L} \hat{g}(x, y) D_k(y, y') (\phi(y, z) - \phi(y', z)). \tag{4.18}$$

We fix for the moment k and x. If $t > 0$, and

$$\sum\nolimits_{\mathbf{j}} \mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z) \leq t/2, \tag{4.19}$$

and we have

$$\begin{aligned} \mathbb{P}\left(\left\| \xi_x^{(k)} (B_3(k), \cdot) \right\|_1 \geq t \right) &\leq \mathbb{P}\left(\left| \sum\nolimits_{\mathbf{j}} \left(\xi_{x,\mathbf{j}}^{(k)}(z) - \mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z) \right) \right| \geq t/2 \right) \\ &\leq K(d,k) \max_{1 \leq \ell \leq K(d,k)} \mathbb{P}\left(\left| \sum\nolimits_{\mathbf{j} \in J_{\ell,k}} \left(\xi_{x,\mathbf{j}}^{(k)}(z) - \mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z) \right) \right| \geq t/(2K(d,k)) \right). \end{aligned}$$

The random variables $\xi_{x,\mathbf{j}}^{(k)}(z) - \mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z)$, $\mathbf{j} \in J_{\ell,k}$, are independent and centered, due to (4.17), and we are going to estimate their sup-norm. We have by (3.13) that $|\phi(y, z) - \phi(y', z)| \leq Ck (\log L)^{-3} L^{-d}$ for y, y' for which $D_k(y, y') \neq 0$. According to Lemma 3.9 c), we have

$$\hat{g}(x, W_{\mathbf{j},k}) \leq Ck^d \left(1 + \frac{d(x, W_{\mathbf{j},k})}{s(L)} \right)^{-d+2}.$$

Substituting that into (4.18), we get

$$\left\| \xi_{x,\mathbf{j}}^{(k)}(z) \right\|_\infty \leq Ck^{d+1} 8^{-k} \left(1 + \frac{d(x, W_{\mathbf{j},k})}{s(L)} \right)^{-d+2} L^{-d} (\log L)^{-12}.$$

By Hoeffding's inequality (see e.g. [6, (1.23)]), we have for $1 \leq \ell \leq K(d,k)$

$$\mathbb{P}\left(\left|\sum\nolimits_{\mathbf{j}\in J_{\ell,k}}\left(\xi_{x,\mathbf{j}}^{(k)}(z)-\mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z)\right)\right| \geq \frac{2^{-k}L^{-d}}{2K(d,k)(\log L)^{37/4}}\right)$$
$$\leq 2\exp\left[-\frac{1}{C}\frac{(\log L)^{-37/2}}{k^{2d+2}4^{-2k}(\log L)^{-24}\sum_{r=1}^{C(\log L)^3}r^{-d+3}}\right] \leq 2\exp\left[-\frac{1}{C}\frac{(\log L)^{5/2}}{k^{2d+2}4^{-2k}}\right],$$

where we used $d \geq 3$ in the last inequality. The upshot of this estimate is that provided (4.19) holds true with $t = t(k) = 2^{-k}L^{-d}(\log L)^{-37/4}$, we have

$$\sup_x \mathbb{P}\left(\sum\nolimits_{k\geq 1}\left\|\xi_x^{(k)}(B_3)\right\|_1 \geq (\log L)^{-37/4}\right) \leq 2\sum_{k\geq 1}K(d,k)\exp\left[-\frac{1}{C}\frac{(\log L)^{5/2}}{k^{2d+2}4^{-2k}}\right]$$
$$\leq \exp\left[-(\log L)^{17/8}\right],$$

It remains to prove (4.19) with this t. Write

$$\sum_{\mathbf{j}}\mathbb{E}\xi_{x,\mathbf{j}}^{(k)}(z) = \sum_{y\in B_3}\sum_{y'\in V_L}\hat{g}(x,y)\,\mathbb{E}(D_k(y,y'))(\phi(y,z)-\phi(y',z)).$$

For every y, $y' \mapsto \mathbb{E}(D_k(y,y'))$ is a signed measure with total mass 0, which is invariant under lattice isometries. Furthermore

$$\sum_{y'}|\mathbb{E}(D_k(y,y'))| \leq C8^{-k}(\log L)^{-9}.$$

Applying (3.14), we get

$$\left|\sum\nolimits_{y'}\mathbb{E}(D_k(y,y'))(\phi(y,z)-\phi(y',z))\right|$$
$$\leq C8^{-k}(\log L)^{-9}\left(L^{-d-1/4}+\left(\frac{Lk(\log L)^{-3}}{L}\right)^3 L^{-d}\right) \leq C4^{-k}(\log L)^{-18}L^{-d},$$

uniformly in $y \in B_3(k)$, and k. By Lemma 3.9 d), we have

$$\sup_x \sum_{y\in B_3(k)}\hat{g}(x,y) \leq C(\log L)^6.$$

From this (4.19) follows. ■

4.2 The non-linear part, good environment

Proposition 4.4
If L is large enough and $\Psi \in \mathcal{M}_L$, then, with $D_{L,\Psi}(0)$ as in (2.6),

$$\mathbb{P}\left(D_{L,\Psi}(0) \geq (\log L)^{-9}; \mathrm{Good}_L\right) \leq \exp\left[-(\log L)^{17/8}\right].$$

Proof. We recall the abbreviation $\mathrm{Sh}_L \stackrel{\mathrm{def}}{=} \mathrm{Shell}_L\left(r\left(L\right)\right)$, c.f. (3.3). By Proposition 4.3, it suffices to estimate on $G_L \cap \mathrm{Good}_L$ the expression $\|\mathcal{R}_L \hat{\pi}_\Psi\|_1$, c.f. (4.2), where

$$\mathcal{R}_L \hat{\pi}_\Psi \stackrel{\mathrm{def}}{=} \sum_{m=1}^{\infty} \sum_{k_1,\ldots,k_m=0}^{\infty} \left(\hat{g}\Delta^{k_1}\Delta\hat{\pi}1_{V_L}\right) \cdot \ldots \cdot \left(\hat{g}\Delta^{k_n}\Delta\hat{\pi}1_{V_L}\right) \sum_{k=1}^{\infty} \left(\hat{g}\Delta^k \phi\right), \tag{4.20}$$

and $\phi = \pi_L \hat{\pi}_\Psi$. The last factor in the right hand side of (4.20) is $\sum_{k=1}^{\infty} \xi^{(k)}$ of the last section, and therefore, it suffices to show that on Good_L,

$$\sup_x \sum_{k\geq 0} \left\| \left(\hat{g}\Delta^k \Delta\hat{\pi}1_{V_L}\right)(x,\cdot)\right\|_1 \leq 15/16. \tag{4.21}$$

Using the definition of Good_L in the first inequality and Lemma 3.9 d) together with Lemma 4.2 in the second, we get

$$\sup_{y\notin \mathrm{Sh}_L} \|(\Delta\hat{\pi})(y,\cdot)\|_1 \leq C\left(\log L\right)^{-9}, \quad \sum_{k\geq 0} \sup_x \left\|\left(\hat{g}\Delta^k\right)(x,\cdot)\right\|_1 \leq C\left(\log L\right)^6.$$

Therefore, we have

$$\sum_{k\geq 0} \sup_x \left\| \sum\nolimits_{y\notin \mathrm{Sh}_L} \left(\hat{g}\Delta^k\right)(x,y)\left(\Delta\hat{\pi}\right)(y,\cdot)\right\|_1 \leq 1/16,$$

if L is large enough, and in order to prove (4.21) it therefore suffices to prove

$$\sum_{k\geq 0} \sup_x \left\| \sum\nolimits_{y\in \mathrm{Sh}_L} \left(\hat{g}\Delta^k\right)(x,y)\left(\Delta\hat{\pi}1_{V_L}\right)(y,\cdot)\right\|_1 \leq 7/8.$$

As in the proof of proposition (4.3), if $\Delta^k(z,y) > 0$ for $y \in \mathrm{Sh}_L$ then $z \in B_1(k)$. Hence, using (3.18) and Lemma 4.2,

$$\begin{aligned}
&\sum_{k\geq 1} \sup_x \left\| \sum\nolimits_{y\in \mathrm{Sh}_L} \left(\hat{g}\Delta^k\right)(x,y)\left(\Delta\hat{\pi}\right)(y,\cdot)\right\|_1 \leq \sum_{k\geq 1} \sup_x \hat{g}(x, B_1(k)) \sup_{z\in B_1(k)} \left\|\Delta^{k+1}(z,\cdot)\right\|_1 \\
&\leq \sum_{k=2}^{20\log\log L+1} k\left(\frac{1}{8}\right)^k + \sum_{k\geq 20\log\log L+2} (\log L)^6 \left(\frac{1}{8}\right)^k < \frac{1}{8}.
\end{aligned}$$

Therefore, it suffices to prove

$$\sup_x \left\| \sum\nolimits_{y\in \mathrm{Sh}_L} \hat{g}\left(x,y\right)\left(\Delta\hat{\pi}1_{V_L}\right)(y,\cdot)\right\|_1 \leq 3/4. \tag{4.22}$$

From the second part of (3.20) it follows that

$$\sup_{x\in V_L} \hat{g}\left(x, \mathrm{Sh}_L\right) \leq 10/9, \quad \sup_{x\in \mathrm{Sh}_L} \pi_{V_{k_0 r(L)}(x)}\left(x, V_L\right) \leq 1/10.$$

By the third part of (3.20), and the choice $\delta < 1/800 < 1/32$, we get

$$\sup_{x\in \mathrm{Sh}_L} \Pi_{V_{k_0 r(L)}(x)}\left(x, V_L\right) \leq \delta + 17/32 \leq 9/16.$$

Combining that, we get

$$\begin{aligned}
&\sup_x \left\| \sum\nolimits_{y\in \mathrm{Sh}_L} \hat g\left(x,y\right)\left(\Delta\hat\pi 1_{V_L}\right)\left(y,\cdot\right)\right\|_1 \\
\le &\sup_x \sum\nolimits_{y\in \mathrm{Sh}_L} \hat g\left(x,y\right)\Pi_{V_{k_0 r(L)}(y)}\left(y,V_L\right) + \sup_x \sum\nolimits_{y\in \mathrm{Sh}_L} \hat g\left(x,y\right)\pi_{V_{k_0 r(L)}(y)}\left(y,V_L\right) \\
\le &\frac{10}{9}\cdot\frac{9}{16}+\frac{10}{9}\cdot\frac{1}{10}<\frac{3}{4},
\end{aligned}$$

proving (4.22). We conclude that $\sup_{x\in V_L}\left\|\mathcal{R}_L\hat\pi_\Psi\left(x,\cdot\right)\right\|_1 \le C\left(\log L\right)^{-37/4}$ on $G_L\cap \mathrm{Good}_L$. ■

4.3 Green function estimates in a goodified environment

Before proceeding to analyze environments where bad regions are present, we consider first "goodified" transition kernels $\mathrm{gd}\left(\hat\Pi\right)$, c.f. (4.8). We write $\tilde G_L$ for the Green function corresponding to this transition kernel. The goal of this section is to derive some estimates on $\tilde G_L$, which will be useful in handling the event $\left(\mathrm{Good}_L\cup \mathrm{TwoBad}_L\right)^c$.

Recall the range $\rho=\rho_{1,L}$, c.f. (3.7), and consider the collection

$$\mathcal{D}_L=\left\{V_{5\rho(x)}\left(x\right)\ ,x\in V_L\right\}\ . \tag{4.23}$$

Lemma 4.5
There exists a constant c_0 such that for all $D\in\mathcal{D}_L$, $D\cap \mathrm{Shell}_L(L/2)\neq\emptyset$,

$$\tilde G_L(0,D)\le c_0\left[\frac{\mathrm{diam}(D)^{d-2}\left(\max_{y\in D} d_L(y)\vee s(L)\right)}{L^{d-1}}\right]\ . \tag{4.24}$$

Further, there exists a constant $c_1\ge 1$ such that for all $D\in\mathcal{D}_L$,

$$\sup_{y\in V_L}\tilde G_L(y,D)\le c_1\ . \tag{4.25}$$

Proof of Lemma 4.5. We begin by establishing some auxiliary estimates for the unperturbed Green function $\hat g=\hat g_L$. We first show that there is a constant C such that for any $D\in\mathcal{D}_L$,

$$\sup_{y\in V_L}\hat g(y,D)=\sup_{y\in D}\hat g(y,D)\le C\ . \tag{4.26}$$

For D such that $D\cap\mathrm{Shell}_L(2s(L))\neq\emptyset$, the estimate (4.26) is an immediate consequence of parts a) and b) of Lemma 3.9. If $D\cap\mathrm{Shell}_L(2s(L))=\emptyset$, then

$$\max_{y,z\in D}\hat g(y,z)=\max_{y\in D}\hat g(y,y)\le 1+\max_{z\in\partial V_{\gamma s(L)}(x)}\hat g(z,y)\le 1+\frac{C}{s(L)^d}\,,$$

where C depends on γ and the second inequality follows from part c) of Lemma 3.9. Summing over $z\in D$ completes the proof of (4.26).

We next note that, for any $z\in V_L$,

$$\hat g(z,D)\le P_z^{\mathrm{RW}}\left(T_D<\tau_{V_L}\right)\max_{w\in D}\hat g(w,D)\,.$$

Applying (4.26) and Lemma 3.4, we deduce that for some constant C_0,

$$\hat{g}(z,D) \leq C_0 \left[\frac{\operatorname{diam}(D)^{d-2} d_L(z) \max_{y\in D} d_L(y)}{d(z,D)^d} \wedge 1 \right] . \tag{4.27}$$

We now turn to proving (4.25). Write the perturbation expansion

$$\tilde{G}_L(z,D) - \hat{g}(z,D) = \sum_{k\geq 1} \sum_{y,y',w} \hat{g}(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}(w,D) + \mathrm{NL} , \tag{4.28}$$

where NL denotes the nonlinear term in the perturbation expansion, that is

$$\mathrm{NL} = \sum_{m=2}^{\infty} \sum_{k_1,\ldots,k_m=0}^{\infty} \left(\hat{g}_L \Delta^{k_1} \Delta\hat{\pi}\right) \cdot \ldots \cdot \left(\hat{g}_L \Delta^{k_{m-1}} \Delta\hat{\pi}\right) \left(\hat{g}_L \Delta^{k_m} \Delta\hat{g}(\cdot,D)\right) . \tag{4.29}$$

We first handle the linear term in (4.28). Using (4.26), part d) of Lemma 3.9, and Lemma 4.2, we see that in a goodified environment,

$$|\sum_{k\geq 1} \sum_{y,y',w: d_L(y')\geq k_0 r(L)} \hat{g}(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}(w,D)| \leq \frac{C(\log L)^{6-9}}{8^k} , \tag{4.30}$$

and

$$|\sum_{y,y',w} \hat{g}(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}(w,D)| \leq \frac{C(\log L)^6}{8^k} . \tag{4.31}$$

From (4.31) it follows that

$$|\sum_{k\geq 20\log\log L} \sum_{y,y',w} \hat{g}(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}(w,D)| \leq C(\log L)^{-9} . \tag{4.32}$$

On the other hand, if $d_L(y') \leq k_0 r(L)$ and $\Delta^k(y,y') > 0$ then, as in the proof of Proposition 4.3, $d_L(y) \leq (4/3)^k k_0 r(L)$. Using parts a),b) of Lemma 3.9, we get that for $k \leq 20 \log\log L$,

$$|\sum_{y,y',w: d_L(y')\leq k_0 r(L)} \hat{g}(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}(w,D)| \leq Ck(1/8)^k \tag{4.33}$$

Combining (4.30), (4.32) and (4.33), we conclude that

$$\sup_{z\in V_L} \sum_{k\geq 1} \sum_{y,y',w} \hat{g}(z,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}(w,D) \leq C .$$

The term involving NL is handled by recalling that

$$\sup_x \sum_{k\geq 0} \left\| \left(\hat{g}_L \Delta^k \Delta\hat{\pi}\right)(x,\cdot) \right\|_1 \leq 15/16 ,$$

see (4.21). We then conclude, using (4.26), that (4.25) holds.

To prove (4.24), our starting point is the perturbation expansion (4.28). Again, the main contribution is the linear term. From (4.31) one deduces that there exists a constant c_d such that for all L large,

$$\sum_{k\geq c_d \log\log L} \sum_{y,y',w} \hat{g}(0,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}(w,D) \leq \left(\frac{r(L)}{L}\right)^{d-1} . \tag{4.34}$$

We divide the sum in the linear term according to the location of w with respect to D, writing

$$\sum_{y,y',w} \hat{g}(0,y)\Delta^k(y,y')\hat{\pi}(y',w)\hat{g}(w,D) = \sum_{y,y'} \hat{g}(0,y)\Delta^k(y,y') \sum_{j=1}^{2} \sum_{w\in B_j} \hat{\pi}(y',w)\hat{g}(w,D)\,, \tag{4.35}$$

where

$$B_1 = \{w \in V_L : d(w,D) \le L/8\}\,, \quad B_2 = \{w \in V_L : d(w,D) > L/8\}\,.$$

Considering the term involving B_1, for $k < c_d \log\log L$ the summation over y extends over a subset of V_L that is covered by at most Ck^d elements of $\mathcal{D}_L$, all inside $\mathrm{Shell}_L(3L/4)$. Thus, for such k, using (4.27) to bound $\hat{g}(0,y)$, (4.26) to bound $\hat{g}(w,D)$, and Lemma 4.2, we get

$$\sum_{y,y'} \hat{g}(0,y)\Delta^k(y,y') \sum_{w\in B_1} \hat{\pi}(y',w)\hat{g}(w,D) \le C\left(\frac{1+\gamma}{8}\right)^k k^d \frac{\mathrm{diam}(D)^{d-2}\max_{y\in D} d_L(y)}{L^{d-1}}$$

and hence

$$\sum_{k\le c_d \log\log L}\sum_{y,y'} \hat{g}(0,y)\Delta^k(y,y') \sum_{w\in B_1} \hat{\pi}(y',w)\hat{g}(w,D) \le C\frac{\mathrm{diam}(D)^{d-2}\max_{y\in D} d_L(y)}{L^{d-1}}\,. \tag{4.36}$$

The term involving $w \in B_2$ is simpler: indeed, one has in that case that $\hat{g}(w,D)$ satisfies, by (4.27), the required bound, whereas for $k < c_d \log\log L$, using (4.26),

$$\sum_{y:\exists y' \text{ with } \Delta^k(y,y')\hat{\pi}(y',w)>0} \hat{g}(0,y) \le Ck^d\,,$$

yielding

$$\begin{aligned} &\sum_{k\le c_d \log\log L}\sum_{y,y'} \hat{g}(0,y)\Delta^k(y,y') \sum_{w\in B_2} \hat{\pi}(y',w)\hat{g}(w,D) \\ &\le C \sum_{k\le c_d \log\log L} k^d (1/8)^k \frac{\mathrm{diam}(D)^{d-2}\max_{y\in D} d_L(y)}{L^{d-1}}\,. \end{aligned} \tag{4.37}$$

Combining (4.34), (4.36) and (4.37) results in the required control on the linear term in (4.28). The nonlinear term is even simpler and similar to the handling of the nonlinear term when estimating $\hat{g}(z,D)$. ∎

4.4 Presence of bad regions

On $(\mathrm{Good}_L \cup \mathrm{TwoBad}_L)^c$, it is clear that for some $D \in \mathcal{D}_L$, c.f. (4.23), we have

$$B_L \subset D \tag{4.38}$$

We write $\mathrm{Bad}_L(D)$ for the event that $\{B_L \subset D\}$, and $\mathrm{Bad}_L^{(i)}(D)$ for the event that $\left\{B_L^{(i)} \subset D\right\}$, $i = 1,2,3,4$. The main aim of this section is to prove the following.

Proposition 4.6
There exists a $\delta_0 \leq 1/800\bar{c}$ *such that if* $\delta < \delta_0$, *and if* $\mathrm{Cond}\,(L_1, \delta)$ *holds for a given* L_1, *and if* $L \leq L_1 (\log L_1)^2$ *and* $\Psi \in \mathcal{M}_L$, *then, for* $i = 1, 2, 3, 4$,

$$\sup_{D \in \mathcal{D}_L} \mathbb{P}\left(D_{L,\Psi}(0) \geq (\log L)^{-9+\frac{9(i-1)}{4}},\ \mathrm{Bad}_L^{(i)}(D) \right) \leq \frac{\exp\left[-(\log L)^2 \right]}{100}.$$

Proof of Proposition 4.6. We start with the case when D is "not near" the boundary, meaning that $D \subset V_{L/2}$. We write $D = V_{5\rho(x_0)}(x_0) = V_{5\gamma s(L)}(x_0)$. By Lemma 3.9 c), we can find a constant K (not depending on L, x_0), such that for any point $x \notin \widetilde{D} \stackrel{\mathrm{def}}{=} V_{5K\gamma s(L)}(x_0)$, and all L large, one has $\hat{g}(x, D) \leq 1/10$. We modify now the transition probabilities $\hat{\Pi}, \hat{\pi}$ slightly, when starting in $x \in D$, by defining

$$\widetilde{\Pi}(x, \cdot) \stackrel{\mathrm{def}}{=} \begin{cases} \mathrm{ex}_{\widetilde{D}}\left(x, \cdot; \hat{\Pi}\right) & \text{for } x \in D \\ \hat{\Pi}(x, \cdot) & \text{for } x \notin D \end{cases}, \tag{4.39}$$

and similarly we define $\widetilde{\pi}$. (Remark that this destroys somewhat the symmetry, when $x \neq x_0$, but this is no problem below). Clearly, these transition probabilities have the same exit distribution from V_L as the one used before. If we write $\widetilde{g}$ for the Green's function on V_L of $\widetilde{\pi}$, we have $\widetilde{g}(x, y) = \hat{g}(x, y)$ for $y \notin \widetilde{D}$, and all x, whereas $\widetilde{g}(x, y) \leq \hat{g}(x, y)$ for $y \in \widetilde{D}$. In particular, we have

$$\sup_{x \notin \widetilde{D}} \widetilde{g}(x, D) \leq 1/10. \tag{4.40}$$

Writing down the perturbation expansion (4.2) using the kernels $\tilde{\Pi}$ and $\tilde{\pi}$, we have

$$([\Pi - \pi]\, \hat{\pi}_\Psi) = \sum_{m=1}^{\infty} \sum_{k_1, \ldots, k_m = 0}^{\infty} \left(\widetilde{g} \Delta^{k_1} \Delta \hat{\pi} \right) \cdot \ldots \cdot \left(\widetilde{g} \Delta^{k_{m-1}} \Delta \hat{\pi} \right) \left(\widetilde{g} \Delta^{k_m} \Delta \phi \right),$$

where Δ now uses the modified transitions, that is $\Delta(x, y) = \tilde{\Pi}(x, y) - \tilde{\pi}(x, y)$, but remark that for $x \notin D$, $\Delta(x, \cdot)$ is the same as before, and that always $\Delta \widetilde{\pi} = \Delta \hat{\pi}$. Also, ϕ is modified accordingly.

We first estimate the part with $m = 1$. In anticipation of what follows, we consider an arbitrary starting point $x \in V_L$. Put $k = k_1 + 1$. The part of the sum

$$\sum_y \sum_{x_1, \ldots, x_k} \widetilde{g}(x, x_1)\, \Delta(x_1, x_2) \cdot \ldots \cdot \Delta(x_k, y)\, \phi(y, \cdot)$$

where all $x_j \notin D$, is estimated in Section 4.1, and the probability that it exceeds $(\log L)^{-9}/3$ is bounded by $\exp\left[-(\log L)^2\right]/100$. If an $x_j \in D$, then the sum over x_{j+1} extends only to points outside $\widetilde{D}$, and therefore, the sum over $x_{j+1}, x_{j+2}, \ldots, x_{j+K}$ is running only over points outside D. Therefore

$$\sup_{x_j \in D} \sum_{x_{j+1}, \ldots x_{j+K}} \left| \Delta(x_j, x_{j+1}) \cdot \ldots \cdot \Delta(x_{j+K}, x_{j+K+1}) \right| \leq 2\delta^K. \tag{4.41}$$

Further, let j denote the smallest index such that $x_j \in D$. Let

$$\mathcal{X}_j := \{x_1 : \Delta(x_1, x_2) \cdots \Delta(x_{j-1}, x_j)\} > 0\,.$$

Then $\max_{x_1\in\mathcal{X}_j} d(x_1,D)\le 5j\gamma s(L)$. For $j<(\log L)^2$ it follows that $\mathcal{X}_j\subset V_{L-s(L)}$ and therefore, by (3.19), $\max_{x\in V_L}\tilde{g}(x,\mathcal{X}_j)\le Cj^d$. Thus,

$$\left|\sum\nolimits_{x_1,\dots,x_j}\widetilde{g}\left(x,x_1\right)\Delta\left(x_1,x_2\right)\cdots\Delta\left(x_{j-1},x_j\right)\right|\le C\delta^{j-1}j^d\,. \tag{4.42}$$

On the other hand, for $j\ge(\log L)^2$ one has (recalling that $x_i\notin D$ for $i<j$, and applying part d) of Lemma 3.9 together with Lemma 4.2),

$$|\sum_{x_1,\dots,x_j}\widetilde{g}\left(x,x_1\right)\Delta\left(x_1,x_2\right)\cdots\Delta\left(x_{j-1},x_j\right)|\le C(1/8)^j(\log L)^6\,.$$

Therefore, using (4.41),

$$\sum_{j=1}^{\infty}\left|\sum_{x_1,\dots,x_{j-1}\notin D,x_j\in D}\widetilde{g}\left(x,x_1\right)\Delta\left(x_1,x_2\right)\cdots\Delta\left(x_{j-1},x_j\right)\right|\le C\,.$$

If $x_k\notin D$, then on the event $B_L\subset D$, using part a) of Proposition 3.8, it holds that $\left\|\sum_y\Delta\left(x_k,y\right)\phi\left(y,\cdot\right)\right\|_1\le C\left(\log L\right)^{-12}$. On the other hand, if $x_k\in D$, then

$$\left\|\sum\nolimits_y\Delta\left(x_k,y\right)\phi\left(y,\cdot\right)\right\|_1\le C\gamma K\left(\log L\right)^{-12+2.25i}\,. \tag{4.43}$$

Combining all the above, we conclude that for some constant c_2 it holds that

$$|\sum_{y,z}\sum_{x_1,\dots,x_k}\widetilde{g}\left(x,x_1\right)\Delta\left(x_1,x_2\right)\cdot\ldots\cdot\Delta\left(x_k,y\right)\phi\left(y,z\right)|\le c_2\gamma K(\log L)^{-12+2.25i}\,.$$

It follows that

$$\left\|{\sum}'_{x_1,\dots,x_k}\widetilde{g}\left(0,x_1\right)\Delta\left(x_1,x_2\right)\cdot\ldots\cdot\left(\Delta\phi\right)\left(x_k,\cdot\right)\right\|_1\le\left(\log L\right)^{-11.5+2.25i}\,, \tag{4.44}$$

where $\sum'$ denotes summation where at least one x_j is in D. (We note that for $i=1,2,3$, one does not need to use the K-enlargement and modification of the transition probabilities, as a factor δ is caught for each $\|\Delta\|_1$.)

The case $m\ge2$ is handled with an evident modification of the above procedure, using the estimate (4.40). Indeed, let $D'=\{z\in V_L: d(z,\tilde{D})\le2\gamma s(L)\}$. A repeat of the previous argument shows that

$$\sup_x\sum_{k=4}^{\infty}\sum_{x_k}|\sum_{\substack{x_1,\dots,x_{k-1}:\\ \exists j\le k,x_j\in D'}}\widetilde{g}\left(x,x_1\right)\Delta\left(x_1,x_2\right)\cdot\ldots\cdot\Delta\left(x_{k-2},x_{k-1}\right)\hat{\pi}\left(x_{k-1},x_k\right)|\le C\delta$$

while

$$\sup_x\sum_{x_3}|\sum_{\substack{x_1,x_2:\\ \exists j\le3,x_j\in D'}}\widetilde{g}\left(x,x_1\right)\Delta\left(x_1,x_2\right)\hat{\pi}\left(x_2,x_3\right)|\le\left\{\begin{array}{ll}\frac{2}{10}\,, & x\notin D'\\ C\,, & x\in D'\,,\end{array}\right.$$

and, by the computation in Section 4.2, c.f. (4.21),

$$\sup_x\sum_{k=3}^{\infty}\sum_{x_k\notin D'}|\sum_{\substack{x_1,\dots,x_{k-1}:\\ x_j\notin D'}}\widetilde{g}\left(x,x_1\right)\Delta\left(x_1,x_2\right)\cdot\ldots\cdot\Delta\left(x_{k-2},x_{k-1}\right)\hat{\pi}\left(x_{k-1},x_k\right)|\le\frac{15}{16}\,.$$

Hence, we conclude that always,

$$\sup_x \sum_{k\geq 0} \left\| \left(\hat{g}\Delta^k \Delta\hat{\pi}\right)(x,\cdot)\right\|_1 \leq C, \tag{4.45}$$

and for all δ small,

$$\sup_x \sum_{k_1,k_2\geq 0} \left\| \left(\hat{g}\Delta^{k_1}\Delta\hat{\pi}\right)\left(\hat{g}\Delta^{k_2}\Delta\hat{\pi}\right)(x,\cdot)\right\|_1 \leq \frac{16}{17}\,. \tag{4.46}$$

Together with the computation for $m = 1$, c.f. (4.44) when D' is visited, and Proposition 4.3 when it is not, this completes the proof of Proposition 4.6 in case $D \subset V_{L/2}$.

We next turn to $D \cap \mathrm{Shell}_L(L/2) \neq \emptyset$. Recall the Green function $\tilde{G}_L$ of the goodified environment, introduced above Lemma 4.5. Let Π_L^g denote the exit distribution Π_L from V_L with the environment replaced by the goodified environment. Let $\Delta^g = 1_D(\Pi_L - \Pi_L^g)$. The perturbation expansion (3.2) then gives

$$[\Pi_L - \Pi_L^g](z) = \sum \tilde{G}_L(0,y)\Delta^g(y,y')\Pi_L(y',z)\,,$$

and thus, using part a) of Lemma 4.5 in the second inequality,

$$\|\Pi_L - \Pi_L^g\|_1 \leq 2\tilde{G}_L(0,D) \leq C\frac{s(L)^{d-2}}{L^{d-2}} \leq C(\log L)^{3(2-d)}\,, \tag{4.47}$$

This completes the proof in case $i = 4$ (and also $i = 1,2,3$ if $d \geq 5$, although we do not use this fact).

Consider next the case $i = 1,2,3$ (and $d = 3,4$). Rewrite the perturbation expansion as

$$[\Pi_L - \Pi_L^g](z) = \sum_{k\geq 1}\sum_y \tilde{G}_L\left(\Delta^g\right)^k(0,y)\left(\hat{\Pi}\tilde{G}_L\Delta^g\Pi_L^g\right)(y,z) \tag{4.48}$$

In particular, using Lemma 4.2 and part b) of Lemma 4.5,

$$\begin{aligned}\|\Pi_L - \Pi_L^g\|_1 &\leq C\tilde{G}_L(0,D)\sum_{k\geq 1}(1/8)^k(\log L)^{-9+2.25i}\sup_{y'\in V_L}\tilde{G}_L(y',D)\\ &\leq C(\log L)^{3(2-d)}(\log L)^{-9+2.25i} \leq (\log L)^{-11.5+2.25i}\,.\end{aligned} \tag{4.49}$$

∎

5 The non-smoothed exit estimate

The aim of this section is to prove the following.

Proposition 5.1
There exists $0 < \delta_0 \leq 1/2$ such that for $\delta \leq \delta_0$, there exist $L_0(\delta)$ and $\varepsilon_0(\delta)$ such that if $L_1 \geq L_0$ and $\varepsilon \leq \varepsilon_0$, then $\mathrm{Cond}(L_1,\delta)$, and $L \leq L_1(\log L_1)^2$ imply

$$\mathbb{P}\left(\|\Pi_L(0,\cdot) - \pi_L(0,\cdot)\|_1 \geq \delta\right) \leq \frac{1}{10}\exp\left[-(\log L)^2\right]\,.$$

Before starting the proof, we provide a sketch of the main idea. As with the smoothed estimates, the starting point is the perturbation expansion (4.2). In contrast to the proof in Section 4, however, no smoothing is provided by the kernel $\hat{\pi}_\Psi$, and hence the lack of control of the exit measure in the last step of the coarse-graining scheme $\mathcal{S}_1$ does not allow one to propagate the estimate on $D_{L,0}(0)$. This is why we need to work with the scheme $\mathcal{S}_2$ introduced in Definition 2.1. Using $\mathcal{S}_2$ means that we refine the coarse graining scale up to the boundary, and when carrying out the perturbation expansion, less smoothing is gained from the coarse graining for steps near the boundary. The drawback of $\mathcal{S}_2$ is that the presence of many bad regions close to the boundary is unavoidable. We will however show that these regions are rather sparse, so that with high enough probability, the RWRE avoids the bad regions. As in Section 4.4, this will be achieved by an appropriate estimate on the Green function in a "goodified" environment.

Proof of Proposition 5.1. We use the coarse graining scheme $\mathcal{S}_2$ from Definition 2.1, but we stick to the notations before, so $\hat{\pi} = \hat{\pi}_{\mathcal{S}_2,L}$, etc. Using $\mathcal{S}_2$ means that we refine the coarsening scale up to the boundary. In particular, $h_L(x) = \gamma d_L(x)$ for all x with $d_L(x) \leq s(L)/2$, and $\hat{\pi}(x,\cdot)$ is obtained by averaging exit distributions from balls with radii between $\gamma d_L(x)$ and $2\gamma d_L(x)$ (γ from (3.4)). If $d_L(x) < 1/2\gamma$, then there is no coarsening at all, and $\hat{\pi}(x,\cdot) = p^{\mathrm{RW}}(x,\cdot)$.

To handle the presence of many bad regions near the boundary, we introduce the layers

$$\Lambda_j \stackrel{\text{def}}{=} \mathrm{Shell}_L\left(2^{j-1}, 2^j\right), \tag{5.1}$$

for $j = 1, \ldots, J_1(L) \stackrel{\text{def}}{=} \left[\frac{\log r(L)}{\log 2}\right] + 1$, so that

$$\mathrm{Shell}_L(r(L)) \subset \bigcup\nolimits_{j \leq J_1(L)} \Lambda_j \subset \mathrm{Shell}_L(2r(L)). \tag{5.2}$$

We subdivide each Λ_j into subsets $D_1^{(j)}, D_2^{(j)}, \ldots, D_{N_j}^{(j)}$ of diameter $\leq \sqrt{d}2^j$, where

$$C^{-1}\left(L2^{-j}\right)^{d-1} \leq N_j \leq C\left(L2^{-j}\right)^{d-1}. \tag{5.3}$$

The collection of these subsets is denoted by $\mathcal{L}_j$. $\mathcal{L}_j$ is split into disjoint $\mathcal{L}_j^{(1)}, \ldots, \mathcal{L}_j^{(R)}$, such that for any m one has

$$d(D, D') > 5\gamma 2^j, \ \forall D, D' \in \mathcal{L}_j^{(m)}, \tag{5.4}$$

$$N_j^{(m)} \stackrel{\text{def}}{=} \left|\mathcal{L}_j^{(m)}\right| \geq N_j/2R. \tag{5.5}$$

We can do that in such a way that $R \in \mathbb{N}$ depends only on the dimension d (recall that γ is fixed by (3.4) once the dimension is fixed).

For $x \in \cup_{j=1}^{J_1(L)} \Lambda_j$, we modify the definition of Good_L in order to adapt it to the smoothing scheme $\mathcal{S}_2$. Thus, we set $\hat{B}_L^{(4)}$ to consist of the union (over $j = 1, \ldots, J_1(L)$) of points $x \in \Lambda_j$ which have the property that $D_{\gamma d_L(x),0}(x) \geq \delta$. We also write $\widehat{\mathrm{Good}_L} = V_L \setminus \hat{B}_L^{(4)}$.

If $B \in \mathcal{L}_j$, we write $\mathrm{Bad}(B)$ for the event $\left\{B \not\subset \widehat{\mathrm{Good}_L}\right\}$. Remark that

$$\mathbb{P}(\mathrm{Bad}(B)) \leq C2^{(d+1)j} \exp\left[-\frac{10}{13}\log^2\left(\gamma 2^{j-1}\right)\right] \leq \exp\left[-j^{5/3}\right] \stackrel{\text{def}}{=} p_j. \tag{5.6}$$

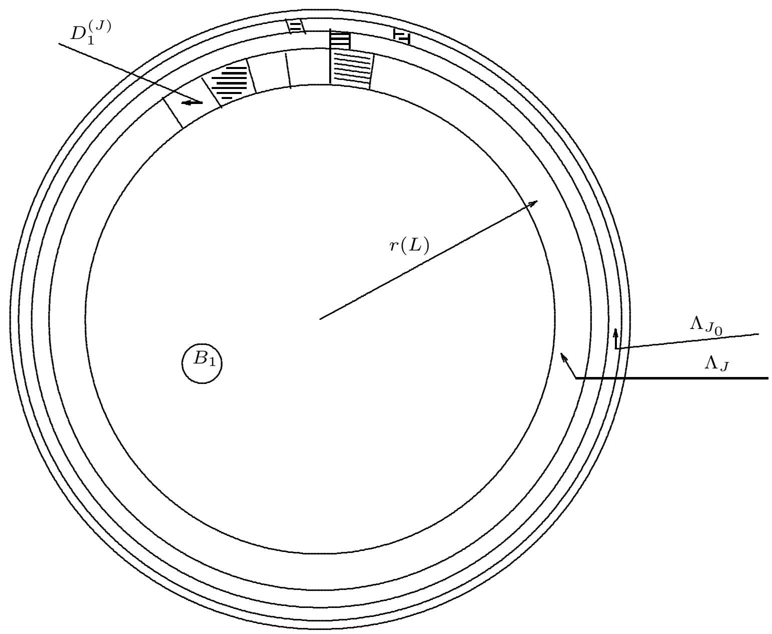

Figure 2: The layers Λ_j. Bad regions (excluding $B_1 \neq \emptyset$) are shaded.

for $j \geq J_0$, J_0 appropriately chosen (depending on d).

We set

$$X_j^{(m)} \stackrel{\text{def}}{=} \sum_{D \in \mathcal{L}_j^{(m)}} 1_{\text{Bad}(D)}, \; X_j \stackrel{\text{def}}{=} \sum_{m=1}^{R} X_j^{(m)}.$$

Due to (5.4), the events $\text{Bad}(D)$, $D \in \mathcal{L}_j^{(m)}$, are independent. Remark that $p_j < j^{-3/2} \leq 1/2$ for all $j \geq 2$. From a standard coin tossing estimate via Chebycheff's inequality, we get

$$\mathbb{P}\left(X_j^{(m)} \geq j^{-3/2} N_j^{(m)}\right) \leq \exp\left[-N_j^{(m)} I\left(j^{-3/2} \mid p_j\right)\right]$$

with $I(x \mid p) \stackrel{\text{def}}{=} x \log(x/p) + (1-x)\log((1-x)/(1-p))$, and

$$I\left(j^{-3/2} \mid p_j\right) \geq -\frac{3}{2} j^{-3/2} \log j + j^{-3/2} j^{5/3} - \log 2 \geq 2Rj^{1/7}$$

for $j \geq J_0$, if J_0 is large enough. Therefore

$$\begin{aligned}\mathbb{P}\left(X_j \geq j^{-3/2} N_j\right) &\leq R \max_{1 \leq m \leq R} \mathbb{P}\left(X_j^{(m)} \geq j^{-3/2} N_j^{(m)}\right) \\ &\leq R \exp\left[-\left(L2^{-j}\right)^{d-1} j^{1/7}\right] \leq R \exp\left[-\frac{1}{C}(\log L)^{20} j^{1/7}\right]\end{aligned}$$

for $J_0 \leq j \leq J_1(L)$, L large enough (implied by L_0 large enough). Using this, we get for $L \geq L_0$,

$$\sum_{J_0 \leq j \leq J_1(L)} \mathbb{P}\left(X_j \geq j^{-3/2} N_j\right) \leq \frac{1}{20} \exp\left[-(\log L)^2\right],$$

increasing further J_0 and L_0 if necessary. Setting

$$\mathrm{ManyBad}_L \stackrel{\mathrm{def}}{=} \bigcup_{J_0 \le j \le J_1(L)} \left\{ X_j \ge j^{-3/2} N_j \right\} \cup \mathrm{TwoBad}_L,$$

we get

$$\begin{aligned}\mathbb{P}\left(\mathrm{ManyBad}_L\right) &\le \frac{1}{20} \exp\left[-\left(\log L\right)^2\right] + \exp\left[-1.2\left(\log L\right)^2\right] \\ &\le \frac{1}{10} \exp\left[-\left(\log L\right)^2\right],\end{aligned} \tag{5.7}$$

for all L large enough (note that the choice of J_0 and L_0 made above depended on the dimension only). We now choose $\varepsilon_0 > 0$ small enough such that for $\varepsilon \le \varepsilon_0$, one has $X_j = 0$, deterministically, for $j < J_0$.

We will show now that if $\omega \notin \mathrm{ManyBad}_L$, then $\|\Pi_L - \pi_L\|_1 \le \delta$, which together with (5.7), will prove Proposition 5.1. Toward this end, distinguish between two (disjoint) bad regions $B_1, B_2 \subset V_L$. We set $\widetilde{B}_L \stackrel{\mathrm{def}}{=} B_L \backslash \mathrm{Sh}_L$, (B_L is as in (4.5)). Set

$$B_2' \stackrel{\mathrm{def}}{=} \bigcup \left\{ D_i^{(j)} : \omega \in \mathrm{Bad}\left(D_i^{(j)}\right),\ j = 1, \ldots, J_1(L);\ i \le N_j \right\}. \tag{5.8}$$

On the complement of TwoBad_L there exists x_0 with $d_L(x_0) > r\left(L\right)$, such that $\widetilde{B}_L \subset V_{5\rho(x_0)}\left(x_0\right)$. (See (4.38). There is some ambiguity in choosing x_0, but this of no importance. In particular, x_0 is arbitrary if $\widetilde{B}_L = \emptyset$.) If $|x_0| \le L/2$, we define $B_1 \stackrel{\mathrm{def}}{=} V_{5\rho(x_0)}\left(x_0\right) = V_{5\gamma s(L)}\left(x_0\right)$, and $B_2 \stackrel{\mathrm{def}}{=} B_2'$. If $|x_0| > L/2$, we put $B_1 \stackrel{\mathrm{def}}{=} \emptyset$, and $B_2 \stackrel{\mathrm{def}}{=} B_2' \cup V_{5\rho(x_0)}\left(x_0\right)$. Of course, if $\widetilde{B}_L = \emptyset$, then $B_1 \stackrel{\mathrm{def}}{=} \emptyset$, and $B_2 \stackrel{\mathrm{def}}{=} B_2'$. Remark that B_1 and B_2 are disjoint. We put $B \stackrel{\mathrm{def}}{=} B_1 \cup B_2$, and $G \stackrel{\mathrm{def}}{=} V_L \backslash B$.

In case $B_1 = V_{5\gamma s(L)}\left(x_0\right)$, $|x_0| \le L/2$, we use the same (slight) modification of $\hat{\Pi}\left(y,\cdot\right)$, $\hat{\pi}\left(y,\cdot\right)$ for $y \in V_{5\gamma s(L)}\left(x_0\right)$ as used in Section 4.4, i.e. we replace $\hat{\pi}, \hat{\Pi}$ by $\widetilde{\pi}, \widetilde{\Pi}$ as defined in (4.39), but we retain the ˆ-notation for convenience.

We use a slight modification of the perturbation expansion (3.2). Again with $\Delta \stackrel{\mathrm{def}}{=} \hat{\Pi} - \hat{\pi}$, we have

$$\Pi_L = \pi_L + \hat{g} 1_B \Delta \Pi_L + \hat{g} 1_G \Delta \Pi_L.$$

Set $\gamma_k \stackrel{\mathrm{def}}{=} \hat{g}\left(1_G \Delta\right)^k$. Then

$$\begin{aligned}\gamma_k \Pi_L &= \hat{g}\left(1_G \Delta\right)^k \Pi_L \\ &= \hat{g}\left(1_G \Delta\right)^k \pi_L + \hat{g}\left(1_G \Delta\right)^k \hat{g} \Delta \Pi_L \\ &= \hat{g}\left(1_G \Delta\right)^k \pi_L + \hat{g}\left(1_G \Delta\right)^k 1_B \Delta \Pi_L + \hat{g}\left(1_G \Delta\right)^k \hat{\pi} \hat{g} \Delta \Pi_L + \gamma_{k+1} \Pi_L\end{aligned}$$

Therefore, iterating, we get

$$\begin{aligned}\Pi_L &= \pi_L + \hat{g} \sum_{k=0}^{\infty} \left(1_G \Delta\right)^k 1_B \Delta \Pi_L + \hat{g} \sum_{k=1}^{\infty} \left(1_G \Delta\right)^k \hat{\pi} \hat{g} \Delta \Pi_L + \hat{g} \sum_{k=1}^{\infty} \left(1_G \Delta\right)^k \pi_L \\ &= \pi_L + \hat{g} \overline{\Gamma} 1_B \Delta \Pi_L + \hat{g} \Gamma \hat{\pi} \Pi_L.\end{aligned}$$

where $\Gamma \stackrel{\mathrm{def}}{=} \sum_{k=1}^{\infty} (1_G\Delta)^k$, $\overline{\Gamma} \stackrel{\mathrm{def}}{=} I + \Gamma$. With the partition $B = B_1 \cup B_2$, we get, setting $\Xi_1 \stackrel{\mathrm{def}}{=} \hat{g}\overline{\Gamma}1_{B_1}\Delta$, $\Xi_2 \stackrel{\mathrm{def}}{=} \hat{g}\overline{\Gamma}1_{B_2}\Delta$,

$$\Pi_L = \pi_L + \Xi_1\Pi_L + \Xi_2\Pi_L + \hat{g}\Gamma\hat{\pi}\Pi_L,$$

and by induction on $m \in \mathbb{N}$, replacing successively Π_L in the second summand

$$\Pi_L - \pi_L = \left(\sum_{r=1}^{m} \Xi_1^r\right)\pi_L + \left(\sum_{r=0}^{m} \Xi_1^r\right)\Xi_2\Pi_L + \left(\sum_{r=0}^{m} \Xi_1^r\right)\hat{g}\Gamma\hat{\pi}\Pi_L + \Xi_1^{m+1}\Pi_L$$

i.e. with $m \to \infty$

$$\begin{aligned}\Pi_L - \pi_L &= \sum_{r=1}^{\infty} \Xi_1^r\pi_L + \left(\sum_{r=0}^{\infty} \Xi_1^r\right)\Xi_2\Pi_L + \left(\sum_{r=0}^{\infty} \Xi_1^r\right)\hat{g}\Gamma\hat{\pi}\Pi_L \\ &:= A_1 + A_2 + A_3\,.\end{aligned} \tag{5.9}$$

For $D \subset V_L$, we write

$$U_k(D) \stackrel{\mathrm{def}}{=} \left\{y \in V_L : \exists x \in D \text{ with } \Delta^k(y,x) > 0\right\}.$$

We now prove that each of the three parts A_1, A_2, A_3 is bounded by $\delta/3$.
First summand A_1 : This does not involve the bad regions near the boundary, and we can apply the estimates from Section 4.4. There is nothing to prove if $B_1 = \emptyset$, so we assume $B_1 = V_{5\gamma s(L)}(x_0)$, $|x_0| \le L/2$. We have

$$\sup_{x\in V_L}\left|\hat{g}(1_G\Delta)^k(x,B_1)\right| \le \sup_{x\in V_L}\delta^k\hat{g}(x,U_k(B_1)) \le C\delta^k k^d, \tag{5.10}$$

where the second inequality is due to part c) of Lemma 3.9, and therefore,

$$\sum_{k=0}^{\infty}\left\|\hat{g}(1_G\Delta)^k 1_{B_1}\right\|_1 \le C. \tag{5.11}$$

In the same way, we obtain, with K from Section 4.4,

$$\sum_{k=0}^{\infty}\sup_{x\notin V_{5K\gamma s(L)}}\left\|\hat{g}(1_G\Delta)^k 1_{B_1}(x,\cdot)\right\|_1 \le \frac{1}{2}, \tag{5.12}$$

by using (5.10) for $k \ge 1$, and (4.40) for $k = 0$. Furthermore,

$$\begin{aligned}\|\Xi_1\pi_L\|_1 &\le \sum_{k=0}^{\infty}\left\|\hat{g}(1_G\Delta)^k 1_{B_1}\Delta\pi_L\right\|_1 \le C\sum_{k=0}^{\infty}\|\hat{g}(\cdot,U_k(B_1))\|_\infty 2^{-k}\sup_{x\in B_1}\|\Delta\pi_L(x,\cdot)\|_1 \\ &\le C\sum_{k=0}^{\infty}k^d 2^{-k}\sup_{x\in B_1}\|\Delta\pi_L(x,\cdot)\|_1 \le C(\log L)^{-3}.\end{aligned} \tag{5.13}$$

Using these inequalities, we get $\|A_1\|_1 \le C(\log L)^{-3} \le C(\log L_0)^{-3} \le \delta/3$ by choosing $L_0(\delta)$ large enough: When estimating $\|\Xi_1^r\pi_L\|_1$ for $r \ge 2$, we use (5.11) for the first factor Ξ_1, (5.13) for the last $\Xi_1\pi_L$, and (5.12) for the middle Ξ_1^{r-2}. The

point is that $(1_{B_1}\Delta)(x,y)$ is $\neq 0$ only if $y \notin V_{5K\gamma s(L)}(x_0)$, and so we can use (5.12) for this part.

Second summand A_2: We drop here the Π_L-factor, using the trivial estimate $\|\Pi_L(x,\cdot)\|_1 \leq 2$. If $r=0$, one has to estimate $\|\Xi_2(0,\cdot)\|_1$ where B_2 consists of the bad regions in the layers $\mathcal{L}_j$, and the possible one bad ball from $\widetilde{B}_L$ which is outside $V_{L/3}$. In case $r \geq 1$, when $B_1 \neq \emptyset$, we have $B_2 = B_2'$, which is at distance $\geq L/3$ from B_1. Therefore, in case $r=0$, we have to estimate

$$\left\| \hat{g}\,(1_G\Delta)^k\, 1_{B_2}(0,\cdot) \right\|_1 \tag{5.14}$$

(the last Δ is of no help, and we drop it), and in case $r \geq 1$, using (5.11) and (5.12)

$$C2^{-r} \sup_{|x|\leq 2L/3} \left\| \hat{g}\,(1_G\Delta)^k\, 1_{B_2}(x,\cdot) \right\|_1,$$

but in this case, we have $B_2 \subset \mathrm{Shell}_L(2r(L))$. The estimate of the second case is entirely similar to the estimate of (5.14), and we therefore provide the details only of the proof of the latter.

We split the parts coming from the different bad regions. For a bad region $D_i^{(j)}$ in layer $\mathcal{L}_j$, we have

$$\left\| \hat{g}\,(1_G\Delta)^k\, 1_{D_i^{(j)}}(0,\cdot) \right\|_1 \leq C2^{-k}\hat{g}\left(0, U_k\left(D_i^{(j)}\right)\right).$$

It suffices to estimate $\hat{g}\left(0, U_k\left(D_i^{(j)}\right)\right)$ very crudely. Points in $U_k\left(D_i^{(j)}\right)$ are at distance of at most $r_{j,k} = 2^j(1-2\gamma)^{-k}$ from $D_i^{(j)}$. We first consider k's only such that $V_L \setminus \mathrm{Shell}_L(s(L))$ is not touched, which is the case if $k \leq 20\log\log L$ (L large enough). Then, for some y with $0.5r_{j,k} \leq d_L(y) \leq 2.5r_{j,k}$, $U_k\left(D_i^{(j)}\right) \subset B(y, r_{j,k}) =: B_{j,k}$. Applying Lemma 3.4, we see that the probability that simple random walk started at the origin hits $B_{j,k}$ before τ_L is bounded above by

$$C2^{(d-1)j}(1-2\gamma)^{-(d-1)k}L^{-d+1} \leq C2^{(d-1)j}\left(\frac{3}{2}\right)^k L^{-d+1},$$

where in the last inequality, we have used the definition of γ, c.f. (3.4). Combined with Lemma 3.9 b), we conclude that for any r such that $\Lambda_r \cap U_k\left(D_i^{(j)}\right) \neq \emptyset$, it holds that

$$\hat{g}\left(0, U_k\left(D_i^{(j)}\right) \cap \Lambda_r\right) \leq C2^{(d-1)j}\left(\frac{3}{2}\right)^k L^{-d+1}.$$

The number of layers r touched is bounded by $2(1+k)$, and thus we conclude that

$$\hat{g}\left(0, U_k\left(D_i^{(j)}\right)\right) \leq C(1+k)\,2^{(d-1)j}\left(\frac{3}{2}\right)^k L^{-d+1}.$$

Therefore, using $\omega \notin \bigcup_{J_0\leq j\leq J_1(L)}\left\{X_j \geq j^{-3/2}N_j\right\}$, we have the estimates

$$\sum_{k\leq 10\log\log L} \left\| \hat{g}\,(1_G\Delta)^k\, 1_{B_2'\cap\Lambda_j}(0,\cdot) \right\|_1 \leq Cj^{-3/2},$$

$$\sum_{k\leq 10\log\log L} \left\| \hat{g}\,(1_G\Delta)^k\, 1_{B_2'}(0,\cdot) \right\|_1 \leq CJ_0^{-1/2}.$$

For the sum over $k > 20 \log\log L$, we simply estimate $\hat{g}\left(0, U_k\left(B_2'\right)\right) \leq \hat{g}\left(0, V_L\right) \leq C\left(\log L\right)^6$ and we therefore get

$$\begin{aligned}\sum_k \left\| \hat{g}\left(1_G \Delta\right)^k 1_{B_2 \cap \mathrm{Shell}_L(r(L))}\left(0, \cdot\right)\right\|_1 &\leq C\left(J_0^{-1/2} + \left(\log L\right)^6 2^{-20 \log\log L}\right) \qquad (5.15)\\ &\leq C\left(J_0^{-1/2} + \left(\log L\right)^{-7}\right) \leq \delta/6\end{aligned}$$

for all $L \geq L_0$, by choosing $J_0 = J_0(\delta)$ and $L_0 = L_0(\delta)$ large enough (again, depending only on d and δ).

It remains to add the part of B_2 outside B_2'. This is (contained in) a ball $V_{5\gamma\rho(x_0)}\left(x_0\right)$ with $|x_0| > L/2$.

$$\hat{g}\left(0, U_k\left(V_{5\gamma\rho(x_0)}\left(x_0\right)\right)\right) \leq \hat{g}\left(0, U_k\left(V_{5\gamma s(L)}\left(x_0\right)\right)\right) \leq \hat{g}\left(0, V_{(5+2k)\gamma s(L)}\left(x_0\right)\right).$$

As $|x_0| \geq L/2$, we have $V_{(5+2k)\gamma s(L)}\left(x_0\right) \cap V_{L/3} = \emptyset$ provided $k \leq \left(\log L\right)^3/C$, and $V_{(5+2k)\gamma s(L)}\left(x_0\right)$ can be covered by $\leq Ck^d$ balls $V_{s(L)}\left(y\right), |y| \geq L/3$. By Lemma 3.4, one has $\hat{g}\left(0, V_{s(L)}\left(y\right)\right) \leq C\left(\log L\right)^{-3}$. (This remains true also if $V_{s(L)}\left(y\right)$ intersects $\mathrm{Shell}_L\left(s\left(L\right)\right)$, as is easily checked). Therefore, for $k \leq \left(\log L\right)^3/C$, we have

$$\hat{g}\left(0, U_k\left(V_{5\gamma\rho(x_0)}\left(x_0\right)\right)\right) \leq Ck^d \left(\log L\right)^{-3},$$

and therefore,

$$\begin{aligned}&\sum_k \left\| \hat{g}\left(1_G \Delta\right)^k 1_{V_{5\gamma\rho(x_0)}(x_0)}\left(0, \cdot\right)\right\|_1\\ \leq &C \sum_{k \leq (\log L)^3/C} 2^{-k} k^d \left(\log L\right)^{-3} + C \sum_{k > (\log L)^3/C} 2^{-k} \left(\log L\right)^6 \leq \delta/6,\end{aligned}$$

provided L_0 is large enough. Combining this with (5.15) proves $\left\|A_2\right\|_1 \leq \delta/3$.

Third summand A_3. By the same argument as in the discussion of A_2, it suffices to consider $r = 0$, and we drop Π_L. Then,

$$\begin{aligned}&\sum_{k \geq 1} \left\| \sum_{x \notin \mathrm{Shell}_L(r(L))} \hat{g}\left(1_G \Delta\right)^{k-1}\left(0, x\right)\left(1_G \Delta \hat{\pi}\right)\left(x, \cdot\right)\right\|_1\\ &\leq \sum_{k \geq 1}^{\infty} 2^{-k+1} \hat{g}\left(0, V_L\right) \sup_{x \notin \mathrm{Shell}_L(r(L))} \left\|1_G \Delta \hat{\pi}\left(x, \cdot\right)\right\|_1 \qquad (5.16)\\ &\leq C\left(\log L\right)^{-3} \leq \delta/9\end{aligned}$$

if L_0 is large enough. For $J_0 \leq j \leq J_1\left(L\right)$,

$$\begin{aligned}\left\| \sum_{x \in \Lambda_j} \hat{g}\left(1_G \Delta\right)^{k-1}\left(0, x\right)\left(1_G \Delta \hat{\pi}\right)\left(x, \cdot\right)\right\|_1 &\leq 2^{-k+1} \hat{g}\left(0, U_k\left(\Lambda_j\right)\right) \sup_{x \in \Lambda_j} \left\|1_G \Delta \hat{\pi}\left(x, \cdot\right)\right\|_1\\ &\leq Cj^{-9} 2^{-k+1} \hat{g}\left(0, U_k\left(\Lambda_j\right)\right),\end{aligned}$$

and it is evident from part b) of Lemma 3.9 that $\sum_{k \geq 1} 2^{-k+1} \hat{g}\left(0, U_k\left(\Lambda_j\right)\right) \leq C$. Therefore,

$$\sum_{k \geq 1} \left\| \sum_{J_0 \leq j \leq J_1(L)} \sum_{x \in \Lambda_j} \hat{g}\left(1_G \Delta\right)^{k-1}\left(0, x\right)\left(1_G \Delta \hat{\pi}\right)\left(x, \cdot\right)\right\|_1 \leq C\left(J_0\right)^{-8} \leq \delta/9, \qquad (5.17)$$

if J_0 is chosen large enough (again, independently of ε_0!). On the other hand, putting $\hat{\Lambda} \stackrel{\text{def}}{=} \bigcup_{j \leq J_0} \Lambda_j$,

$$\begin{aligned}
&\sum_{k \geq 1} \left\| \sum\nolimits_{x \in \hat{\Lambda}} \hat{g}\left(1_G \Delta\right)^{k-1}(0, x)\left(1_G \Delta \hat{\pi}\right)(x, \cdot) \right\|_1 \\
&\leq C \sum_{k \geq 1} 2^{-k+1} \hat{g}\left(0, U_k\left(\hat{\Lambda}\right)\right) \sup_{x \in \hat{\Lambda}} \|\Delta(x, \cdot)\|_1 \leq C\left(J_0\right) \sup_{x \in \hat{\Lambda}} \|\Delta(x, \cdot)\|_1 \leq \delta / 9
\end{aligned}$$

if $\varepsilon \leq \varepsilon_0(\delta)$ and $\varepsilon_0(\delta)$ is taken small enough. Combining this with (5.16) and (5.17) proves $\|A_3\|_1 \leq \delta/3$. Substituting the estimate on $\|A_i\|_1$, $i = 1, 2, 3$, into (5.9) and using (5.7) completes the proof of Proposition 5.1. ■

6 Proof of Proposition 2.3

We just have to collect the estimates we have obtained so far. We take δ_0 small enough as in the conclusion of Propositions 4.6 and 5.1, and for $\delta \leq \delta_0$, we choose L_0 large enough, also according to these propositions.

For $L_1 \geq L_0$ we assume $\operatorname{Cond}(\delta, L_1)$, and take and $L \leq L_1 (\log L_1)^2$. For $i = 1, 2, 3$, and $\Psi \in \mathcal{M}_L$, we have according to Lemma 4.1 and Proposition 4.4

$$\begin{aligned}
b_i(L, \Psi, \delta) &\leq \mathbb{P}\left(D_{L,\Psi}(0) > (\log L)^{-11.25-2.25i}\right) \\
&\leq \mathbb{P}\left(D_{L,\Psi}(0) > (\log L)^{-11.25-2.25i}, (\mathrm{TwoBad}_L)^c \cap (\mathrm{Good}_L)^c\right) \\
&+ \mathbb{P}\left(D_{L,\Psi}(0) > (\log L)^{-9}, \mathrm{TwoBad}_L \cap \mathrm{Good}_L\right) + \mathbb{P}(\mathrm{TwoBad}_L) \\
&\leq \mathbb{P}\left(D_{L,\Psi}(0) > (\log L)^{-11.25-2.25i}, (\mathrm{TwoBad}_L)^c \cap (\mathrm{Good}_L)^c\right) \\
&+ \exp\left[-1.2 (\log L)^2\right] + \exp\left[-(\log L)^{17/8}\right].
\end{aligned}$$

We therefore only have to estimate the first summand. Using the notation of Sections 4.3 and 4.4,

$$\begin{aligned}
&\mathbb{P}\left(D_{L,\Psi}(0) > (\log L)^{-11.25-2.25i}, (\mathrm{TwoBad}_L)^c \cap (\mathrm{Good}_L)^c\right) \\
&\leq \sum_{D \in \mathcal{D}_L} \sum_j \mathbb{P}\left(\left\|\left([\Pi_{V_L} - \pi_{V_L}] \hat{\pi}_\Psi\right)(0, \cdot)\right\|_1 \geq (\log L)^{-11.25+2.25i},\ \mathrm{Bad}_L^{(j)}(D)\right) \\
&\leq \sum_{D \in \mathcal{D}_L} \sum_{j \leq i} \mathbb{P}\left(\left\|\left([\Pi_{V_L} - \pi_{V_L}] \hat{\pi}_\Psi\right)(0, \cdot)\right\|_1 \geq (\log L)^{-11.25+2.25j},\ \mathrm{Bad}_L^{(j)}(D)\right) \\
&\quad + \sum_{D \in \mathcal{D}_L} \sum_{j > i} \mathbb{P}\left(\mathrm{Bad}_L^{(j)}(D)\right) \\
&\leq \frac{4|\mathcal{D}_L|}{100} \exp\left[-(\log L)^2\right] + |\mathcal{D}_L| \exp\left[-[1 - (4 - i - 1)/13]\left(\log \frac{L}{(\log L)^{10}}\right)^2\right] \\
&\leq \frac{1}{8} \exp\left[-[1 - (4 - i)/13] (\log L)^2\right].
\end{aligned}$$

Combining these estimates, we get for $i = 1, 2, 3$ and L large enough,

$$b_i(L, \Psi, \delta) \leq \frac{1}{4} \exp\left[-[1 - (4 - i)/13] (\log L)^2\right].$$

For $i = 4$, we have

$$b_4(L, \Psi, \delta) \leq \mathbb{P}\left(D_{L,\Psi}(0) > (\log L)^{-2.25}\right) + \mathbb{P}\left(\|\Pi_L(0,\cdot) - \pi_L(0,\cdot)\|_1 \geq \delta\right).$$

The second summand is estimated by Proposition 5.1, and the first in the same way as the b_i, $i \leq 3$. This completes the proof of Proposition 2.3.

7 Proof of Theorem 2.5

The proof is based on a modification of the computations in Section 5. We begin by an auxiliary definition. In what follows, c_d is a constant large enough so as to satisfy

$$\log c_d > 4d. \tag{7.1}$$

For any $x \in V_L$ and random walk $\{X_n\}$ with $X_0 = x$, set $\eta(x) = \min\{n > 0 : |X_n - x| > d_L(x)\}$. We fix δ small enough such that $\delta < \delta_0$ and

$$\bar{c}_d \stackrel{\text{def}}{=} \max_{y \in V_L} p^{\text{RW},y}(\tau_L > \eta(y)) + \delta < 1 \tag{7.2}$$

for all L large enough (this is possible by the invariance principle for simple random walk). We then chose ε_0 so as the conclusion of Theorem 2.4 is satisfied with this value of δ. For $x \in \Lambda_j$, set $U_1(x) = \partial V_{\gamma d_L(x)}(x)$, and inductively, for $k \geq 2$, set $U_k(x) = \cup_{y \in U_{k-1}(x)} \partial V_{c_d^k 2^j}(y)$.

Definition 7.1
A point $x \in \Lambda_j$ is K-good if $D_{\gamma d_L(x),0}(x) \leq \delta$ and, for any $k = 1, \ldots, K$, for all $y \in U_k(x)$, $D_{c_d^{k+1} 2^j,0}(y) \leq \delta$.

We note that there exists some constant C depending only on d such that for all $x \in \Lambda_j$,

$$\mathbb{P}(x \text{ is not } K\text{-good}) \leq \exp\left(-\frac{10}{13}(\log(\gamma 2^{j-1}))^2\right) + \sum_{k=1}^{K} C(c_d^k 2^j)^d \exp\left(-\frac{10}{13}(\log(c_d^k 2^j))^2\right). \tag{7.3}$$

For $J > J_0$, set

$$\tilde{B}^{(4)}_{L,J,K} \stackrel{\text{def}}{=} \left(\cup_{j=J+1}^{J_1(L)} \{x \in \Lambda_j : D_{\gamma d_L(x),0}(x) \geq \delta\}\right) \bigcup \left(\{x \in \Lambda_J : x \text{ is not } K\text{-good}\}\right),$$

and $\widetilde{\text{Good}}_{L,J,K} \stackrel{\text{def}}{=} V_L \setminus \tilde{B}^{(4)}_{L,J,K}$.

If $B \in \mathcal{L}_j$, $j \geq J$, we write $\widetilde{\text{Bad}}_{L,J,K}(B)$ for the event $\left\{B \not\subset \widetilde{\text{Good}}_{L,J,K}\right\}$. Remark that for $J > J_1(L)$, $\widetilde{\text{Bad}}_{L,J,K}(B) = \text{Bad}(B)$. By combining the computation in (5.6) with (7.3), and using our choice for the value of c_d, we get that there exists a $J_2 \geq J_0$ such that for all $J > J_2$, all K, and all L large enough,

$$\mathbb{P}\left(\widetilde{\text{Bad}}_{L,J,K}(B)\right) \leq p_J. \tag{7.4}$$

We set next

$$\tilde{X}_j \stackrel{\text{def}}{=} \sum_{D \in \mathcal{L}_j} 1_{\widetilde{\text{Bad}}_{L,J,K}(D)}$$

and

$$\widetilde{\mathrm{ManyBad}}_{L,J,K} \stackrel{\mathrm{def}}{=} \bigcup_{J_0(\gamma)\le j\le J(L)} \left\{\tilde{X}_j \ge j^{-3/2}N_j\right\} \cup \mathrm{TwoBad}_L \ .$$

Arguing as in the computation leading to (5.7) (except that $\mathcal{L}_J$ is divided into more sets to achieve independence, and the number of such sets depends on K), we conclude that for each J,K there is an $L_2 = L_2(J,K)$ such that for all $L > L_2(J,K)$,

$$\mathbb{P}\left(\widetilde{\mathrm{ManyBad}}_{L,J,K}\right) \le \frac{1}{10}\exp\left[-(\log L)^2\right] . \tag{7.5}$$

Next, replace B_2' in (5.8) by considering in the union there only $j \in [J, J_1(L)]$, and replacing $\mathrm{Bad}\left(D_i^{(j)}\right)$ by $\widetilde{\mathrm{Bad}}_{J,K,L}\left(D_i^{(j)}\right)$ (note that this influences only the layers Λ_j with $j \le J$). We rewrite the expansion (5.9)

$$\begin{aligned}\Pi_L - \pi_L &= \sum_{r=1}^{\infty}\Xi_1^r\pi_L + \left(\sum_{r=0}^{\infty}\Xi_1^r\right)\Xi_2\Pi_L + \left(\sum_{r=0}^{\infty}\Xi_1^r\right)\hat{g}\Gamma\hat{\pi}\Pi_L \\ &:= A_1 + A_2 + A_3 \,.\end{aligned} \tag{7.6}$$

except that now $\tilde{B}_2'$ is used instead of B_2' in the definition of Ξ_2. By repeating the computation leading to (5.13) and (5.15) (using (7.5) instead of (5.7) in the latter), we conclude that

$$\|A_1\| + \|A_2\| \le CJ^{-1/2} \,, \tag{7.7}$$

for L large. To analyze A_3, we write, with obvious notation, $A_3 = \sum_{r=0}^{\infty} A_3^{(r)}$, and argue as in Section 5 that it is enough to consider $A_3^{(0)}$. We then write

$$\begin{aligned}A_3^{(0)}(\cdot) \quad &= \quad \sum_{k\ge1}\sum_{x\notin \mathrm{Shell}_L(2^J)} \hat{g}\left(1_G\Delta\right)^{k-1}(0,x)\left(1_G\Delta\hat{\pi}\right)(x,z)\,\Pi_L(z,\cdot) \\ &+ \sum_{k\ge1}\sum_{x\in \mathrm{Shell}_L(2^J)} \hat{g}\left(1_G\Delta\right)^{k-1}(0,x)\left(1_G\Delta\hat{\pi}\right)(x,z)\,\Pi_L(z,\cdot) = A_3^{(0),1} + A_3^{(0),2} \,.\end{aligned} \tag{7.8}$$

We already have from Section 5 that on the event $\left(\widetilde{\mathrm{ManyBad}}_{L,J,K}\right)^c$, it holds that $\|A_3^{(0),1}\|_1 \le CJ^{-1/2}$. Note that all the estimates so far held for any large fixed J,K, as long as L is large enough (large enough depending on the choice of J,K). It remains only to analyze $\|A_3^{(0,1)}\hat{\pi}_s\|_1$ for s independent of L, and when doing that, we can choose K in any way that does not depend on L. As a preliminary step in the choice of K, we have the following lemma (recall the constant $\bar{c}_d$, c.f. (7.2)):

Lemma 7.2
For all J,K, and any K-good $x \in \Lambda_J$, it holds that for all L large enough,

$$\sum_{y:|x-y|>c_d^{K+2}2^J} |\Delta\hat{\pi}\Pi_L(x,y)| \le (\bar{c}_d)^K \tag{7.9}$$

Proof of Lemma 7.2. The proof is similar to the argument in Lemma 3.1. Consider a RWRE X_n started at $y \in U_1(x)$. Let $\eta_1 = \min\{n : X_n \in \partial V_{c_d 2^J}(y)\}$, and, for

$k=2,\ldots,K$, define successively $\eta_k(y)=\min\{n>\eta_{k-1}(y): X_n\in\partial V_{c_d^k 2^J}(X_{\eta_{k-1}}))\}$. The sum in (7.9) is then bounded above by

$$\max_{y\in U_1(x)} p_\omega^y(\tau_L>\eta_K(y)) \le \prod_{k=1}^K \max_{y\in U_k(x)} p_\omega^y(\tau_L>\eta(y))\,. \tag{7.10}$$

Since x is K-good, we have that for $y\in U_k(x)$, $p_\omega^y(\tau_L>\eta_k(y))\le\delta+p^{\mathrm{RW},y}(\tau_L>\eta(y))\le\bar c_d$. Substituting in (7.10), the lemma follows. ■

We next recall, c.f. Lemma A.4, that

$$\sup_{x_1,x_2:|x_1-x_2|\le D}\|\hat\pi_s(x_1,\cdot)-\hat\pi_s(x_2,\cdot)\|\le \frac{CD\log s}{s}\,.$$

In particular, for any fixed K,J with $J>J_2$, using (7.9), and the fact that $\sum_z A_3^{(0),1}\hat\pi_s(z)=0$, we get

$$\|A_3^{(0),1}\hat\pi_s\|_1\le\delta(\bar c_d)^K+C\frac{c_d^{K+1}\log s}{s}\,.$$

and thus

$$\|A_3\hat\pi_s\|\le\delta(\bar c_d)^K+C\frac{(c_d)^{K+1}2^J\log s}{s}+CJ^{-1/2}\,. \tag{7.11}$$

Combining (7.11) and (7.7), we conclude that on the event $\left(\widetilde{\mathrm{ManyBad}}_{L,J,K}\right)^c$,

$$D_{L,\Psi_s}(0)\le\delta(\bar c_d)^K+C\frac{2(c_d)^{K+1}2^J\log s}{s}+CJ^{-1/2}\,.$$

Choosing J large such that $CJ^{-1/2}<\delta/3$ and K large enough such that $(\bar c_d)^K<\delta/3$, and s large enough such that $2C(c_d)^{K+1}2^J\log s/s<\delta/3$, and using (7.5), it follows that
$\limsup_{L\to\infty}L^r b(L,\Psi_s,\delta)=0$. This completes the proof of the theorem.

A Proofs of the random walk results

We begin by stating and proving some auxiliary estimates. If $A\subset\subset\mathbb{Z}^d$, $x\in A$, $y\in\partial A$, then by time reversibility of simple random walk and transience, one gets

$$P_x\left(X_{\tau_A}=y\right)\le C\sum_{y'\in A,\ |y-y'|=1}P_{y'}\left(T_x<\tau_A\right).$$

Throughout this appendix, we write $\tau\stackrel{\mathrm{def}}{=}\tau_{V_L}$. Since we do not deal with the RWRE in this appendix, we consistently write P_x for P_x^{RW}.

Lemma A.1
Let $x\in V_L,\ y\in\partial V_L$. Then, for some $\bar c_1\ge1$,

$$P_x\left(X_\tau=y\right)\le\bar c_1 d_L\left(x\right)^{-d+1}.$$

Proof. Let $r \stackrel{\text{def}}{=} d_L(x)$. We may assume that $r \geq 4$. Put $r' \stackrel{\text{def}}{=} [r/2] - 1$. Then $V_{r'}(x) \subset V_{L-r'}$. If y' is any neighbor of y in V_L then, by part c) of Lemma 3.3,

$$P_{y'}\left(T_{\partial V_{r'}(x)} < \tau\right) \leq P_{y'}\left(T_{V_{L-r'}} < \tau\right) \leq \frac{C}{r}.$$

Furthermore uniformly in $z \in \partial V_{r'}(x)$,

$$P_z(T_x < \tau) \leq P_z(T_x < \infty) \leq C(r')^{-d+2} \leq Cr^{-d+2}.$$

Using the Markov property and (A.1) proves the claim. ■

Lemma A.2
Let $x \in V_L,\ y \in \partial V_L$ and set $t \stackrel{\text{def}}{=} |x-y|$. Then for some $\bar{c}_2 \geq 1$,

$$P_x(X_\tau = y) \leq \bar{c}_2 \frac{d_L(x)}{t} \sup_{x' \in \partial V_{t/3}(y) \cap V_L} P_{x'}(X_\tau = y).$$

Proof. The bound is evident if $r \stackrel{\text{def}}{=} d_L(x) \geq t/10$. Therefore, we assume $r < t/10$. We choose a point $x' \notin V_L$ such that $V_{t/8}(x') \cap V_L = \emptyset$, and $|x - x'| \leq t/8 + 2r$. Then $x \in V_{t/4}(x')$ and $|x' - y| \geq 3t/4$. Therefore

$$\partial V_{t/4}(x') \cap V_{t/3}(y) = \emptyset. \tag{A.1}$$

A walk starting in x has to reach $\partial V_{t/4}(x')$ before it can reach y, and therefore, if it reaches $V_{t/8}(x')$ before reaching $\partial V_{t/4}(x')$ it exits V_L before it reaches y. After it has reached $\partial V_{t/4}(x') \cap V_L$ it still has to reach $V_{t/3}(y) \cap V_L$ before it can reach y, moving inside V_L. So we get

$$\begin{aligned} P_x(X_\tau = y) &\leq CP_x\left(\tau_{V_{t/4}(x')} < T_{V_{t/8}(x')}\right) \sup_{z \in \partial V_{t/4}(x') \cap V_L} P_z(X_\tau = y) \\ &\leq \frac{Cr}{t} \sup_{z \in \partial V_{t/4}(x') \cap V_L} P_z(X_\tau = y) \leq \frac{Cr}{t} \sup_{z \in \partial V_{t/3}(y) \cap V_L} P_z(X_\tau = y), \end{aligned}$$

the second inequality using Lemma 3.3 c). ■

Lemma A.3
With x, y, t as above, and $\bar{c}_1, \bar{c}_2$ from the previous lemmas,

$$P_x(X_\tau = y) \leq \bar{c}_1 \bar{c}_2^d 3^{(d-1)^2 + (d-1)} \frac{d_L(x)}{t^d}.$$

Proof. Put $\eta \stackrel{\text{def}}{=} 3^{-d+1}\bar{c}_2^{-1}$, and set $\bar{K} \stackrel{\text{def}}{=} \bar{c}_1 \eta^{-d+1} = \bar{c}_1 \bar{c}_2^{d-1} 3^{(d-1)^2}$. Using Lemma A.2, it suffices to prove

$$\sup_{x \in \partial V_r(y) \cap V_L} P_x(X_\tau = y) \leq \bar{K} r^{-d+1}. \tag{A.2}$$

As $\bar{K} \geq 9^{(d-1)}$, there is nothing to prove if $r \leq 9$. Assume that we have proved (A.2) for all $r \leq r_0$, and assume $r_0 < r \leq 2r_0$. Then for $d_L(x) > \eta r$, we have by Lemma A.1 that

$$P_x(X_\tau = y) \leq \bar{c}_1 \eta^{-d+1} r^{-d+1} = \bar{K} r^{-d+1},$$

and for $d_L(x) \leq \eta r$, by Lemma A.2 and the fact that $r/3 \leq r_0$,

$$P_x(X_\tau = y) \leq \bar{c}_2 \eta \bar{K} \left(\frac{r}{3}\right)^{-d+1} = \bar{K} r^{-d+1}.$$

Therefore, the lemma is proved by induction. ■

Proof of Lemma 3.4. If $|x-y| \leq d_L(y)/2$, then $d_L(x) \geq d_L(y)/2$, and in this case, we can simply use part c) of Lemma 3.3 to conclude that

$$P_x\left(T_{V_a(y)} < \tau\right) \leq P_x\left(T_{V_a(y)} < \infty\right) \leq C\left(\frac{a}{|x-y|}\right)^{d-2} \leq C\frac{a^{d-2} d_L(y)\, d_L(x)}{|x-y|^d}.$$

Therefore, we may assume $|x-y| > d_L(y)/2$. Furthermore, it suffices to consider the case $1 \leq a \leq d_L(y)/5$, simply because for $d_L(y)/5 < a \leq 5d_L(y)$, we get an upper bound with replacing a by $d_L(y)/5$. Assume that we have proved the bound for $a = d_L(y)/5$. Then we get for $a < d_L(y)/5$

$$P_x\left(T_{V_a(y)} < \tau\right) \leq C\frac{d_L(y)^{d-1} d_L(x)}{|x-y|^d}\left(\frac{a}{d_L(y)}\right)^{d-2} \leq C\frac{a^{d-2} d_L(y)\, d_L(x)}{|x-y|^d}.$$

We therefore see that it suffices to prove the bound for $a = d_L(y)/5$.

Let $y' \in \partial V_L$ be a point closest to y. There exists $\delta > 0$, such that

$$\inf_{x' \in V_a(y)} P_{x'}(X_\tau \in V_a(y')) \geq \delta.$$

Evidently, $\inf_{z \in V_a(y') \cap \partial V_L} |x-z| \geq |x-y|/2$, and therefore, by Lemma A.3,

$$\sup_{z \in V_a(y') \cap \partial V_L} P_x(X_\tau = z) \leq C\frac{d_L(x)}{|x-y|^d}.$$

Consequently

$$\begin{aligned}
\frac{d_L(x)\, a^{d-1}}{|x-y|^d} &\geq \frac{1}{C} P_x(X_\tau \in V_a(y')) \geq \frac{1}{C} P_x\left(X_\tau \in V_a(y'),\ T_{V_a(y)} < \tau\right) \\
&= \frac{1}{C}\sum_{x' \in V_a(y)} P_x\left(T_{V_a(y)} < \tau,\ X_{T_{V_a(y)}} = x'\right) P_{x'}(X_\tau \in V_a(y')) \\
&\geq \frac{\delta}{C} P_x\left(T_{V_a(y)} < \tau\right).
\end{aligned}$$

This proves the claim. ■ Before presenting the proofs of Lemmas 3.6 and 3.7, we state and prove some additional auxiliary estimates. We define the Brownian analogue $\hat{\pi}_\Psi^{\mathrm{BM}}$ of $\hat{\pi}_\Psi$, c.f. (3.12), by

$$\hat{\pi}_\Psi^{\mathrm{BM}}(x, dz) \stackrel{\mathrm{def}}{=} \int \frac{1}{m_x} \varphi(t/m_x)\, \pi_{C_t}^{\mathrm{BM}}(x, dz) dt.$$

$\hat{\pi}_\Psi^{\mathrm{BM}}(x, dz)$ has a density with respect to Lebesgue measure which, by an abuse of notation, we write as $\hat{\pi}_\Psi^{\mathrm{BM}}(x, z)$.

Lemma A.4
There is a constant C such that for any L large enough, any $\Psi \in \mathcal{M}_L$, any $x, x', z, z' \in \mathbb{Z}^d$, it holds that

$$\hat{\pi}_\Psi(x,z) \leq CL^{-d}, \quad \hat{\pi}_\Psi^{\mathrm{BM}}(x,z) \leq CL^{-d}. \tag{A.3}$$

$$|\hat{\pi}_\Psi(x,z) - \hat{\pi}_\Psi(x',z)| \leq C|x-x'|L^{-(d+1)} \log L, \tag{A.4}$$

$$|\hat{\pi}_\Psi^{\mathrm{BM}}(x,z) - \hat{\pi}_\Psi^{\mathrm{BM}}(x',z)| \leq C|x-x'|L^{-(d+1)} \log L, \tag{A.5}$$

$$|\hat{\pi}_\Psi(x,z) - \hat{\pi}_\Psi(x,z')| \leq C|z-z'|L^{-(d+1)} \log L, \tag{A.6}$$

$$|\hat{\pi}_\Psi^{\mathrm{BM}}(x,z) - \hat{\pi}_\Psi^{\mathrm{BM}}(x,z')| \leq C|x-x'|L^{-(d+1)} \log L. \tag{A.7}$$

Further, for $1 < a < b < 2$, and $aL \leq |x-z| \leq bL$,

$$\hat{\pi}_\Psi(x,z) \geq C(a,b)^{-1} L^{-d}. \tag{A.8}$$

Proof of Lemma A.4. The estimates (A.4) and (A.8) are immediate from Lemmas 3.3 and 3.5, and the definition of $\hat{\pi}_\Psi$.

We turn to the proof of (A.3) and (A.6). It clearly suffices to consider only the cases $|x-x'| = 1$ or $|z-z'| = 1$. Note first that

$$\begin{aligned}
|\hat{\pi}_\Psi(x,z) - \hat{\pi}_\Psi(x',z)| &= \left[1 - \frac{m_x}{m_{x'}}\right] \hat{\pi}_\Psi(x,z) \\
&+ \frac{1}{m_{x'}} \int_{\mathbb{R}^+} \left[\varphi\left(\frac{t}{m_x}\right) - \varphi\left(\frac{t}{m_{x'}}\right)\right] \pi_{V_t(x)}(x,z)dt \\
&+ \frac{1}{m_{x'}} \int_{\mathbb{R}^+} \varphi\left(\frac{t}{m_{x'}}\right) [\pi_{V_t(x)}(x,z) - \pi_{V_t(x')}(x',z)]\, dt \stackrel{\mathrm{def}}{=} I_1 + I_2 + I_3.
\end{aligned}$$

Since $\Psi \in \mathcal{M}_L$, it holds that $\left[1 - \frac{m_x}{m_{x'}}\right] \leq CL^{-1}|x-x'|$, and hence, using (A.4), it holds that

$$I_1 \leq CL^{-d} \frac{|x-x'|}{L}. \tag{A.9}$$

Similarly, using the smoothness of φ and the estimates $m_{x'} \geq L/2$ and $\pi_{V_t(x)}(x,z) \leq CL^{1-d}$, see Lemma 3.3 a), one gets

$$I_2 \leq CL^{-d} \frac{|x-x'|}{L}. \tag{A.10}$$

By translation invariance of simple random walk, we have that $\pi_{V_r(x)}(x,z) = \pi_{V_r}(0, z-x)$. Thus, both (A.3) and (A.6) will follow if we can show, for $|x-x'| = 1$ and $y = x$ or x', the estimate

$$\left|\int_{\mathbb{R}^+} \varphi\left(\frac{t}{m_y}\right) [\pi_{V_t}(0, z-x) - \pi_{V_t}(0, z-x')]\, dt\right| \leq CL^{-d}. \tag{A.11}$$

Of course, we may assume that $|x-z|$ is of order L. Note that the integration in (A.11) is over the union of two intervals, each of length at most $\sqrt{d}$. Hence, due to the smoothness of φ, (A.11) will follow if we can show that

$$\left|\int_{\mathbb{R}^+} [\pi_{V_t}(0, z-x) - \pi_{V_t}(0, z-x')]\, dt\right| \leq CL^{-d}. \tag{A.12}$$

Let $J \overset{\text{def}}{=} \{t > 0 : x - z \in \partial V_t\}$. J is an interval of length at most $\sqrt{d}$. For $t \in J$, we set

$$t' = t'(t) \overset{\text{def}}{=} \left| x' - t\frac{z - x}{|z - x|} \right|.$$

Evidently, $dt'/dt = 1 + O\left(L^{-1}\right)$, and if we set $J' \overset{\text{def}}{=} \{t > 0 : x' - z \in \partial V_{t'}\}$, then J' is an interval of the same length as J, up to $O\left(L^{-1}\right)$, and further $|J \Delta J'| = O\left(L^{-1}\right)$. Therefore, if we prove

$$\left| \int_{J \cap J'} [\pi_{V_t(x)}(x, z) - \pi_{V_{t'}(x')}(x', z)] dt \right| \leq C L^{-d} \log L, \tag{A.13}$$

the estimate (A.11) will follow. To abbreviate notation, we write V for $V_t(x)$, and V' for $V_{t'}(x')$. A first exit decomposition yields

$$\pi_V(x, z) \leq \pi_{V'}(x, z) + \sum_{y \in V \setminus V'} P_x^{\mathrm{RW}}(T_y < \tau_V)\, \pi_V(y, z). \tag{A.14}$$

We have two simple geometric facts:

- $$\bigcup_{t \in J \cap J'} (V \setminus V') \subset x + \mathrm{Shell}_L(C).$$
- For any $y \in x + \mathrm{Shell}_L(C)$
$$\int_{J \cap J'} 1_{\{y \in V \setminus V'\}} dt \leq C \frac{|y - z|}{L}.$$

Using this together with $\pi_{V'}(x, z) = \pi_{V'}(x', z) + O\left(L^{-d}\right)$, see [5, Theorem 1.7.1], we deduce from (A.14) that

$$\begin{aligned} \int_{J \cap J'} \pi_{V_t(x)}(x, z) dt &\leq \int_{J \cap J'} \pi_{V_{t'}(x')}(x', z) dt + O\left(L^{-d}\right) + C L^{-d} \sum_{y \in x + \mathrm{Shell}_L(C)} |y - z|^{-d} \frac{|y - z|}{L} \\ &\leq \int_{J \cap J'} \pi_{V_{t'}(x')}(x', z) dt + O\left(L^{-d} \log L\right) \end{aligned}$$

The inequality in the opposite direction is proved in the same way. This proves (A.12) and completes the proof of (A.3) and (A.6).

The estimates (A.5) and (A.7) can be obtained either by repeating the argument above, replacing the random walk by Brownian motion, or by applying the Poisson formula [5, (1.43)]. We omit further details. ∎

In order to prove Lemma 3.6 we need also the following technical result:

Lemma A.5
There exists a constant $C = C(\beta, \epsilon)$ such that for any $A \in \partial V_L$, $\beta > 6\epsilon > 0$, $y \in V_L$ with $d(y, \partial V_L) > L^\beta$ and $L > L_0$,

$$\sum_{y' \in A} \pi_L(y, y') \leq \int_{d(y', A) \leq L^\beta} \pi_L^{\mathrm{BM}}(y, dy') \left(1 + \frac{C(\beta, \epsilon))}{L^{\beta - 5\epsilon}}\right) + \frac{C(\beta, \epsilon)}{L^{d+1}}. \tag{A.15}$$

and for any $A' \in \partial C_L$ and $z \in V_L$ with $d(z, \partial C_L) > L^\beta$,

$$\int_{A'} \pi_L^{\mathrm{BM}}(z, dy') \le \sum_{y' : d(y', A) \le L^\beta} \pi_L(z, y') \left(1 + \frac{C(\beta,\epsilon)}{L^{\beta - 5\epsilon}}\right) + \frac{C(\beta,\epsilon)}{L^{d+1}} . \tag{A.16}$$

Finally, for any $x, z \in \mathbb{Z}^d$ and $\Psi \in \mathcal{M}_L$,

$$\left|\hat{\pi}_\Psi(x,z) - \hat{\pi}_\Psi^{\mathrm{BM}}(x,z)\right| \le \frac{C}{L^{d+1/4}}. \tag{A.17}$$

Proof of Lemma A.5. We first prove (A.15). Set $A_\beta = \{y' \in \partial C_L : d(y', A) \le L^\beta\}$. Pick $\epsilon \in (0, \beta)$ and set $L' = L + L^\epsilon$ and $L'' = L + L^{2\epsilon}$. Let A'_β be the image of A_β in $\partial C_{L'}$ under the map $x \mapsto (L'/L)x$. Then, one has (with $\hat{y} = L'y/L$),

$$\int_{A_\beta} \pi_L^{\mathrm{BM}}(y, dy') = \int_{A'_\beta} \pi_{L'}^{\mathrm{BM}}(\hat{y}, dy') . \tag{A.18}$$

Note further, using the Poisson formula [5, (1.43)], that

$$\begin{aligned} \int_{A'_\beta} \pi_{L'}^{\mathrm{BM}}(\hat{y}, dy') &= \int_{A'_\beta} \frac{d\pi_{L'}^{\mathrm{BM}}(\hat{y}, \cdot)}{d\pi_{L'}^{\mathrm{BM}}(y, \cdot)} \pi_{L'}^{\mathrm{BM}}(y, dy') \\ &= \int_{A'_\beta} \frac{\left((L')^2 - |\hat{y}|^2\right)|y' - y|^d}{\left((L')^2 - |y|^2\right)|y' - \hat{y}|^d} \pi_{L'}^{\mathrm{BM}}(y, dy') \end{aligned} \tag{A.19}$$

An explicit computation, using that $|y| \le L - L^\beta$ and that $1 > \beta > \epsilon > 0$, reveals that

$$\left|\log \frac{\left((L')^2 - |\hat{y}|^2\right)|y' - y|^d}{\left((L')^2 - |y|^2\right)|y' - \hat{y}|^d}\right| \le C L^{\epsilon - \beta} .$$

Substituting in (A.19) one finds that

$$\int_{A_\beta} \pi_L^{\mathrm{BM}}(y, dy') \ge \int_{A'_\beta} \pi_{L'}^{\mathrm{BM}}(y, dy') \left(1 - C(\beta,\epsilon) L^{-\beta + 2\epsilon}\right) . \tag{A.20}$$

Recall that π_L^{BM} is unchanged if one replaces the Brownian motion by a Brownian motion of covariance $I_d/\sqrt{d}$. Let W_t^y be such a Brownian motion started at y, and recall that by [11, Corollary 1], there exists a constant C_0 such that for every integer n, one may construct $\{W_t^x\}$ in the same space as $\{X_n\}$ such that

$$P_x\left(\max_{0 \le m \le n} |X_m - W_m^x| > C_0 \log n\right) \le \frac{C_0}{n^{d+1}} . \tag{A.21}$$

Standard estimates involving the maximum of the increments of the Brownian motion, imply that one may construct the Brownian motion W_t^y and the random walk X_n on the same space such that, with

$$D \stackrel{\mathrm{def}}{=} \{ \sup_{0 \le t \le L^{2+\epsilon/100}} \left|X_{[t]} - W_t^y\right| \le 4 C_0 \log L \} ,$$

one has

$$P_y(D^c) \le \frac{2C_0}{n^{d+1}} . \tag{A.22}$$

Set $\tau \stackrel{\text{def}}{=} \min\{n : X_n \in \partial V_L\}$, $\tau' \stackrel{\text{def}}{=} \inf\{t : W^y_t \in \partial C_{L'}\}$, $\tau'' \stackrel{\text{def}}{=} \min\{n : X_n \in \partial V_{L''}\}$, and $B \stackrel{\text{def}}{=} \left\{(\tau' \vee \tau'') \le L^{2+\epsilon/100}\right\}$. Standard estimates imply that if $X_0 = y$ then $P(B^c)$ decays like a stretched exponential, and in particular $P(B^c) \le L^{-d-1}$ for large L. Note that on $D \cap B$, one has that $\tau < \tau' < \tau''$. Now, defining $G'_\beta = \{z \in \mathbb{Z}^d : d(z, (A'_\beta)^c \cap \partial C_L) < 4C_0 \log L\}$, and setting $T_{G'_\beta} = \inf\{n : X_n \in G'_\beta\}$,

$$
\begin{aligned}
P\left(W^y_{\tau'} \in A'_\beta\right) &\ge P_y(X_\tau \in A, W_{\tau'} \in A'_\beta) \\
&\ge P_y(X_\tau \in A, W_{\tau'} \in A'_\beta, B \cap D) - \frac{1}{L^{d+1}} \\
&\ge P_y(X_\tau \in A) - P_y(X_\tau \in A, W_{\tau'} \notin A'_\beta, B \cap D) - \frac{2}{L^{d+1}} \\
&\ge P^{\text{RW}}_y(X_\tau \in A) - P^{\text{RW}}_y(X_\tau \in A, T_{G'_\beta} < \tau'') - \frac{2}{L^{d+1}}
\end{aligned} \tag{A.23}
$$

Using the Markov property, one has

$$
\begin{aligned}
&P^{\text{RW}}_y(X_\tau \in A, T_{G'_\beta} < \tau'') \le P^{\text{RW}}_y(X_\tau \in A) \sup_{z \in A} P^{\text{RW}}_z(T_{G'_\beta} < \tau'') \\
&\le \sup_{z \in A} \sum_{z' \in G'_\beta} P^{\text{RW}}_z(T_{z'} < \tau'') \le \sup_{z \in A} C \sum_{z' \in G'_\beta} \frac{L^{3\epsilon} \log^{d+2} L}{|z' - z|^d} \le C L^{5\epsilon - \beta}\,,
\end{aligned}
$$

where the next to last inequality is due to Lemma 3.4. Substituting in (A.23), one completes the proof of (A.15). The reverse inequality (A.16) is proved similarly.

It remains to prove (A.17). Fix $\alpha = 2/3$, $\beta = 1/3$, and $\epsilon = 1/60$. Note that with $\mathcal{D} = C_{L^\alpha}(z)$, using (A.6),

$$
\hat{\pi}_\Psi(x, z) \le \frac{1}{|\mathcal{D}|} \sum_{z' \in \mathcal{D}} \hat{\pi}_\Psi(x, z') + C L^{-d-1+\alpha} \log L\,. \tag{A.24}
$$

Next, note that

$$
\begin{aligned}
\sum_{z' \in \mathcal{D}} \hat{\pi}_\Psi(x, z') &= \int dt \varphi_{m_x}(t) \sum_{z' \in \mathcal{D}} \pi_{V_t(x)}(x, z') \\
&\le \int dt \varphi_{m_x}(t) \int_{C_{L^\alpha + L^\beta}(z)} \pi^{\text{BM}}_t(x, dz') \left(1 + \frac{C}{L^{\beta - 5\epsilon}}\right) + \frac{C|\mathcal{D}|}{L^{d+1}} \\
&\le \hat{\pi}^{\text{BM}}_\Psi(x, \mathcal{D}) \left(1 + \frac{C}{L^{\beta - 5\epsilon}}\right) + C L^{-d} |C_{L^\alpha + L^\beta}(z) \setminus C_{L^\alpha}(z)| + \frac{C|\mathcal{D}|}{L^{d+1}} \\
&\le |\mathcal{D}| \hat{\pi}^{\text{BM}}_\Psi(x, z) \left(1 + \frac{C}{L^{\beta - 5\epsilon}}\right) + \frac{C|\mathcal{D}| \log L}{L^{d+1-\alpha}} + \frac{C|\mathcal{D}|}{L^{\alpha - \beta - d}}\,.
\end{aligned}
$$

Substituting in (A.24), one gets

$$
\hat{\pi}_\Psi(x, z) \le \hat{\pi}^{\text{BM}}_\Psi(x, z) + C L^{-d-1/4}\,.
$$

The reverse equality is proved similarly. This completes the proof of (A.17) and of the lemma ∎

Proof of Lemma 3.6. Fix $\alpha = 2/3, \beta = 1/3$. Set $\eta \stackrel{\text{def}}{=} d(y, \partial V_L)$, and let $y_1 \in \partial V_L$ be such that $\eta = |y - y_1|$. Consider first $\eta \le L^{\beta + 1/15}$. Then, using (3.9) and (A.4)

in the first inequality and (A.3) in the second,

$$\begin{aligned}\phi_{L,\Psi}(y,z) &\le \sum_{y'\in\partial V_L:|y'-y_1|<L^\alpha} \pi_{V_L}(y,y')\,\hat\pi_\Psi(y',z) + \frac{C\log L}{L^{d+\alpha-\beta}}\\ &\le \hat\pi_\Psi(y_1,z) \sum_{y'\in\partial V_L:|y'-y_1|<L^\alpha} \pi_{V_L}(y,y') + \frac{C}{L^{d+1/5}}\,.\end{aligned}$$

Consequently,

$$\phi_{L,\Psi}(y,z) \le \hat\pi_\Psi(y_1,z) + \frac{C}{L^{d+1/5}}\,.$$

Applying now (3.10) in the first inequality and (A.3) in the second, we conclude that

$$\begin{aligned}\phi_{L,\Psi}(y,z) &\le \hat\pi_\Psi(y_1,z) \int_{y'\in\partial V_L:|y'-y_1|<L^\alpha} \pi_L^{\mathrm{BM}}(y,dy') + \frac{C}{L^{d+1/5}}\\ &\le \int_{y'\in\partial V_L:|y'-y_1|<L^\alpha} \hat\pi_\Psi(y',z)\,\pi_L^{\mathrm{BM}}(y,dy') + \frac{C}{L^{d+1/5}}\,.\end{aligned}$$

An application of (A.17) then implies that for $\eta \le L^{\beta+1/15}$,

$$\phi_{L,\Psi}(y,z) \le \phi_{L,\Psi}^{\mathrm{BM}}(y,z) + CL^{-d-1/5}$$

where, as in our convention, the constant C is uniform in the choice of y,z. The reverse inequality is obtained using the same steps.

Consider next $\eta > L^{\beta+1/15}$. Fix strictly positive constants c_j, $j=1,\ldots,4$, depending on d,α only, and a sequence of disjoint sets $A_i \subset \partial V_L$, $i=1,\ldots,k_L$ with $\cup_{i=1}^{k_L} A_i = \partial V_L$, $c_1 L^{\alpha(d-1)} \le |A_i| \le c_2 L^{\alpha(d-1)}$, $\mathrm{diam}(A_i) \le c_3 L^\alpha$, $d(y_1, \partial A_1 \cap \partial V_L) \ge \mathrm{diam}(A_1)/4$, and $|\partial A_i| \cap \partial V_L \le c_4 L^{\alpha(d-2)}$ (such a collection of "cube-like" A_i can clearly be found). We also set $A_i^\beta = \{y \in \mathbb{R}^d : d(y,A_i) \le L^\beta\}$ and for $i \ge 2$, fix an arbitrary $y_i \in A_i$. We then have

$$\begin{aligned}\phi_{L,\Psi}(y,z) &= \sum_{i=1}^{k_L} \sum_{y'\in A_i} \pi_{V_L}(y,y')\,\hat\pi_\Psi(y',z)\\ &\le \sum_{i=1}^{k_L} \hat\pi_\Psi(y_i,z) \sum_{y'\in A_i} \pi_{V_L}(y,y') + \frac{C\log L}{L^{d+1-\alpha}}\,,\end{aligned}$$

where (A.3) was used in the last inequality. Consequently, using (A.15),

$$\phi_{L,\Psi}(y,z) \le \sum_{i=1}^{k_L} \hat\pi_\Psi(y_i,z) \int_{A_i^\beta} \pi_L^{\mathrm{BM}}(y,dy') \left(1 + \frac{C}{L^{1/4}}\right) + \frac{C}{L^{d+1/5}}\,. \tag{A.25}$$

Let $\{\tilde A_i \subset \partial C_L\}_{i=1}^{k_L}$ be a collection of measurable disjoint sets with $\cup \tilde A_i = \partial C_L$, $\tilde A_1 = A_1^\beta \cap \partial C_L$, and $\tilde A_i \subset A_i^\beta$. Using (3.10) and $d(y,\partial C_L) \ge L^{\beta+1/15}/2$, one gets

$$\int_{A_i^\beta} \pi_L^{\mathrm{BM}}(y,dy') \le \int_{\tilde A_i} \pi_L^{\mathrm{BM}}(y,dy') \left(1 + C\frac{|(A_i^\beta \cap \partial C_L)\setminus \tilde A_i|}{|A_i^\beta \cap \partial C_L|}\right)\,.$$

Substituting in (A.25) we get

$$\phi_{L,\Psi}\left(y,z\right) \leq \sum_{i=1}^{k_L} \hat{\pi}_{\Psi}\left(y_i,z\right) \int_{\tilde{A}_i} \pi_L^{\mathrm{BM}}\left(y,dy'\right) \left(1 + CL^{-1/5}\right) + \frac{C}{L^{d+1/5}}\,.$$

Hence, recalling (A.4), (A.3), and (A.17), we get

$$\begin{aligned}\phi_{L,\Psi}\left(y,z\right) &\leq \sum_{i=1}^{k_L} \int_{\tilde{A}_i} \hat{\pi}_{\Psi}\left(y',z\right) \pi_L^{\mathrm{BM}}\left(y,dy'\right) + \frac{C}{L^{d+1/5}} \\ &\leq \sum_{i=1}^{k_L} \int_{\tilde{A}_i} \hat{\pi}_{\Psi}^{\mathrm{BM}}\left(y',z\right) \pi_L^{\mathrm{BM}}\left(y,dy'\right) + \frac{C}{L^{d+1/5}} = \phi_{L,\Psi}^{\mathrm{BM}}(y,z) + \frac{C}{L^{d+1/5}}\,.\end{aligned}$$

The reverse inequality is obtained by a similar argument. ■

Proof of Lemma 3.7. We write $\pi_t^{\mathrm{BM}}\left(w,z\right)$ as the density on $\partial C_t(w)$ of the measure $\pi_{C_t(w)}^{\mathrm{BM}}\left(w,dz\right)$. Set $g(w,z) = \int \pi_t^{\mathrm{BM}}\left(w,z\right) \varphi_{m_w}\left(t\right) dt$. Then,

$$\phi_{L,\Psi}^{\mathrm{BM}}\left(y,z\right) = \int_{\partial C_L(0)} \pi_{C_L(0)}^{\mathrm{BM}}\left(y,dw\right) g(w,z)\,.$$

For $\bar{z} \in \partial C_1(0)$, set

$$\begin{cases} \frac{1}{2}\Delta_{\bar{y}} u(\bar{y},\bar{z}) = 0, & \bar{y} \in C_1(0)\,, \\ u(\bar{y},\bar{z}) = g(L\bar{y},L\bar{z}), & \bar{y} \in \partial C_1(0)\,. \end{cases}$$

Then, $\phi_{L,\Psi}^{\mathrm{BM}}\left(y,z\right) = u(y/L,\bar{z})$ with $\bar{z} = z/L$ and hence

$$\left|\frac{\partial^i \phi_{L,\Psi}^{\mathrm{BM}}(y,z)}{\partial y^i}\right| = \frac{1}{L^i}\left|\frac{\partial^i u(\bar{y},\bar{z})}{\partial \bar{y}^i}\right|\,. \tag{A.26}$$

Write

$$\|u(\bar{y},\bar{z})\|_k = \sum_{j=0}^{k} \sup_{\bar{y},\bar{z}} \left|\frac{\partial^j u(\bar{y},\bar{z})}{\partial \bar{y}^j}\right|\,.$$

By [4, Theorem 6.3.2],

$$\|u\left(\bar{y},\bar{z}\right)\|_3 \leq C\|g(\bar{w},\bar{z})\|_4\,. \tag{A.27}$$

By the smoothness of φ and the translation invariance and scaling properties of the Brownian motion, and applying [3, Theorem 2.10], one gets that

$$\|g(\bar{w},\bar{z})\|_4 \leq L^{-d}\,.$$

Substituting in (A.27) and using (A.26), the lemma follows. ■

B A local CLT and proof of Lemma 3.9

We need a number of properties for simple random walk, and coarse-grained random walks, which can readily be obtained from known results. We keep L and V_L fixed through this section, and don't emphasize them in the notation. π is π_{V_L}, the exit distribution of simple random walk from V_L. Since the proofs are very similar,

and for concreteness, we prove all results for the smoothing scheme $\mathcal{S}_1$ and only sketch the necessary changes for the scheme $\mathcal{S}_2$. Remark that the coarse graining scale at x, $h_L(x)$, equals $\gamma s(L)$ for $d_L(x) \geq 2s(L)$, and $h_L(x) \leq (\gamma/2)\, s(L)$ for $x \in \mathrm{Shell}_L(r, 2s(L))$. By a slight abuse of notation, we write $\hat{\pi}_m$ for the transition probabilities on $\mathbb{Z}^d$ with the constant in x coarse-graining scheme Ψ_m. We also write $\hat{\pi}_m(x)$ for $\hat{\pi}_m(0,x)$. For $x \in V_{L-2s(L)}$, $\hat{\pi}(x,\cdot) = \hat{\pi}_{\gamma s(L)}(x,\cdot)$ under either $\mathcal{S}_i$, $i = 1, 2$.

Let $m \in \mathbb{R}^+$. $\hat{\pi}_m$ is centered, and the covariances satisfy

$$\sum_x x_i x_j \hat{\pi}_m(x) = \alpha(m)\, \delta_{ij},$$

where for some $0 < \alpha_1 < \alpha_2$

$$\alpha_1 m^2 \leq \alpha(m) \leq \alpha_2 m^2.$$

(It is evident that $\sigma_m^2 \stackrel{\mathrm{def}}{=} \alpha(m)/m^2$ converges as $m \to \infty$.) Using Lemma 3.3 a), one sees that for $1 < a < b < 2$, one has for some δ (which may depend on a, b)

$$\inf_{am \leq |x| \leq bm} \hat{\pi}_m(x) \geq \delta m^{-d}. \tag{B.1}$$

Furthermore, by definition, we have $\hat{\pi}_m(x) = 0$ for $|x| \geq 2m$.

We will also use the following fact, proved in Lemma A.4:

$$|\hat{\pi}_m(x) - \hat{\pi}_m(y)| \leq C m^{-d} \left| \frac{x-y}{m} \right|^{1/15}. \tag{B.2}$$

In what follows, we write $\hat{\pi}_m^{*n}$ for the n-fold convolution of $\hat{\pi}_m$.

Proposition B.1

$$\hat{\pi}_m^{*n}(x) = \frac{1}{(2\pi m^2 \sigma_m^2 n)^{d/2}} \exp\left[-\frac{|x|^2}{2m^2\sigma_m^2 n} \right] + O\left(m^{-d} n^{-(d+2)/2} (\log n)^4 \right)$$

Proof of Proposition B.1. The proof is standard, but we need to keep track of the m-dependence, and we are not aware of a reference for that in the literature. Let

$$\chi_m(z) \stackrel{\mathrm{def}}{=} \sum_x \mathrm{e}^{iz\cdot x/m} \hat{\pi}_m(x),\ z \in B_m \stackrel{\mathrm{def}}{=} [-m\pi, m\pi]^d$$

By Fourier inversion, we have

$$\hat{\pi}_m^{*n}(x) = (2\pi)^{-d} m^{-d} \int_{B_m} \mathrm{e}^{-iz\cdot x/m} \chi_m(z)^n \, dz.$$

We will choose $0 < a < A$, $b > 0$, and $\alpha \in (0,1)$ (not depending on n, m) and split

$$\int_{B_m} \mathrm{e}^{-iz\cdot x/m} \chi_m(z)^n \, dz = \int_{|z| \leq \frac{b\log n}{\sqrt{n}}} + \int_{\frac{b\log n}{\sqrt{n}} < |z| \leq a} + \int_{a < |z| \leq A} + \int_{A < |z| \leq m^\alpha} + \int_{m^\alpha < |z|,\ z \in B_m}$$
$$\stackrel{\mathrm{def}}{=} A_1 + A_2 + A_3 + A_4 + A_5.$$

From Taylor's formula, we get

$$\chi_m(z) = 1 - \frac{|z|^2 \sigma_m^2}{2} + O\left(|z|^4\right),$$

and therefore, for $|z| \le 1/C$,

$$\log \chi_m(z) = -\frac{|z|^2 \sigma_m^2}{2} + O\left(|z|^4\right).$$

From that we get for b sufficiently large and $n \ge C(b)$,

$$\begin{aligned} A_1 &= \left(1 + O\left(\frac{(\log n)^4}{n}\right)\right) \int_{|z| \le \frac{b \log n}{\sqrt{n}}} \exp\left[-i\frac{z \cdot x}{m} - \frac{n|z|^2 \sigma_m^2}{2}\right] dz \\ &= \left(1 + O\left(\frac{(\log n)^4}{n}\right)\right) \int \exp\left[-i\frac{z \cdot x}{m} - \frac{n|z|^2 \sigma_m^2}{2}\right] dz + O\left(n^{-d/2-1}\right) \qquad \text{(B.3)} \\ &= \frac{(2\pi)^{d/2}}{n^{d/2}\sigma_m^d} \exp\left[-\frac{|x|^2}{2m^2\sigma_m^2 n}\right] + O\left(n^{-d/2-1}(\log n)^4\right). \end{aligned}$$

In order to prove the proposition, it therefore suffices to prove that $A_2, \ldots, A_5$ are of order $O\left(n^{-d/2-1}\right)$, uniformly in L.

To handle A_2, we choose a such that $\log \chi_m(z) \le -|z|^2 \sigma_m^2/3$ for $|z| \le a$. Then

$$|A_2| \le \int_{\frac{b \log n}{\sqrt{n}} < |z|} \exp\left[-|z|^2 n\sigma_m^2/3\right] dz = O\left(n^{-d/2-1}\right).$$

if we choose b sufficiently large.

For A_3, we use the following fact, which is an easy consequence of (B.1): for any $a < A$, one has

$$\sup_{m, a \le |z| \le A} |\chi_m(z)| < 1. \qquad \text{(B.4)}$$

Using this, we immediately get

$$|A_3| \le CA^d (1 - 1/C)^n. \qquad \text{(B.5)}$$

We come now to A_4 which is more difficult. First remark that since the coarse graining scheme is isotropic, we only have to consider z-values with all components positive. Put $|z|_\infty \overset{\text{def}}{=} \max(z_1, \ldots, z_d)$. For simplicity, we assume that z_1 is the biggest component of z, so that $|z|_\infty = z_1$. Let $M \overset{\text{def}}{=} [2\pi m/z_1]$, and $K \overset{\text{def}}{=} [(2m+1)/M]$. We may assume that $M < m$ by choosing A large enough. We write

$$\begin{aligned} \chi_m(z) = &\sum_{(x_2, \ldots, x_d)} \exp\left[\frac{i}{m} \sum\nolimits_{s=2}^d x_s z_s\right] \\ &\times \left\{ \sum_{j=1}^K \sum_{x_1 = -m+(j-1)M}^{-m+jM-1} \mathrm{e}^{ix_1 z_1/m} \hat{\pi}_m(x) + \sum_{x_1 = -m+KM}^{m} \mathrm{e}^{ix_1 z_1/m} \hat{\pi}_m(x) \right\}. \end{aligned}$$

In the first summand, inside the x_1-summation, we write for each j separately, $\hat{\pi}_m(x) = \hat{\pi}_m(x) - \hat{\pi}_m(x') + \hat{\pi}_m(x')$, where $x' = (-m + (j-1)M, x_2, \ldots, x_d)$. Then, by (B.2),

$$|\hat{\pi}_m(x) - \hat{\pi}_m(x')| \le Cm^{-d} \left(\frac{x_1 + m - (j-1)M}{m} \right)^{1/15}.$$

Therefore,

$$\left| \sum\nolimits_{x_1=-m+(j-1)M}^{-m+jM-1} \mathrm{e}^{ix_1 z_1/m} \left(\hat{\pi}_m(x) - \hat{\pi}_m(x') \right) \right| \le Cm^{-d+1} \frac{1}{z_1^{16/15}},$$

and therefore,

$$\left| \sum_{j=1}^{K} \sum_{x_1=-m+(j-1)M}^{-m+jM-1} \mathrm{e}^{ix_1 z_1/m} \left(\hat{\pi}_m(x) - \hat{\pi}_m(x') \right) \right| \le Cm^{-d+1} |z|^{-1/15}.$$

Also,

$$\left| \sum_{j=1}^{K} \sum_{x_1=-m+(j-1)M}^{-m+jM-1} \mathrm{e}^{ix_1 z_1/m} \hat{\pi}_m(x') \right| \le K\hat{\pi}_m(x') \left| \frac{1 - \exp[iz_1 M/m]}{1 - \exp[iz_1/m]} \right| \le C|z| m^{-d},$$

$$\left| \sum_{x_1=-m+KM}^{m} \mathrm{e}^{ix_1 z_1/m} \hat{\pi}_m(x) \right| \le m^{-d+1} |z|^{-1}.$$

Therefore, we get the estimate

$$|\chi_m(z)| \le C_1 \left(|z|^{-1/15} + \frac{|z|}{m} \right).$$

From this, we get

$$|A_4| \le C_1^n \int_{A \le |z| \le m^\alpha} \left(|z|^{-1/15} + \frac{|z|}{m} \right)^n dz \le 2^{-n} \tag{B.6}$$

for large enough A and m.

For A_5, we need a slight modification. Let again $z_1 > 0$ be the largest of the z-components. Then we write

$$\hat{\pi}_m(x) = \sum_{y=-m}^{x_1} \left(\hat{\pi}_m(y, x_2, \ldots, x_d) - \hat{\pi}_m(y-1, x_2, \ldots, x_d) \right),$$

$$\begin{aligned} \chi_m(z) = 2i &\sum_{x_2,\ldots,x_d} \exp\left[\frac{i}{m} \sum\nolimits_{s=2}^{d} x_s z_s \right] \\ &\times \sum_{y=-m}^{m} \left(\hat{\pi}_m(y, x_2, \ldots, x_d) - \hat{\pi}_m(y-1, x_2, \ldots, x_d) \right) \\ &\times \frac{\mathrm{e}^{i(z_1/m)(y-1/2)} - \mathrm{e}^{i(z_1/m)(m+1/2)}}{\sin(z_1/2m)}. \end{aligned}$$

Therefore

$$\left|\chi_m(z)\right| \leq C m^{d-1} \frac{m}{z_1} \sum_{y=-m}^{m} \left|\hat{\pi}_m(y, x_2, \ldots, x_d) - \hat{\pi}_m(y-1, x_2, \ldots, x_d)\right| \leq C \frac{m^{14/15}}{|z|},$$

and if $\alpha > 1 - \gamma$

$$\begin{aligned} |A_5| &\leq m^{-d} \int_{m^\alpha \leq |z|} \left|\chi_m(z)\right|^n dz \leq C^n m^{-d} m^{14n/15} \int_{m^\alpha}^{\infty} r^{d-1} r^{-n} dz \qquad \text{(B.7)} \\ &\leq C^n m^{-d} m^{14n/15} m^{\alpha(d-n)} \leq 2^{-n}, \end{aligned}$$

if m and n are large enough. Combining (B.3)-(B.7), Proposition B.1 follows. ∎

We next need a simple large deviation estimate

Lemma B.2
There exists $C > 0$, such that for $|x| \geq 2m$,

$$\hat{\pi}_m^{*n}(x) \leq C m^{-d} \exp\left[-\frac{|x|^2}{Cnm^2}\right].$$

Proof of Lemma B.2. If $|x| \geq r$, then one of the d components of x satisfies $|x_i| \geq r/\sqrt{d}$. By rotational symmetry, we get

$$\sum_{x:|x|\geq r} \hat{\pi}_m^{*n}(x) = dP\left(\left|\sum\nolimits_{j=1}^{n} \xi_j\right| \geq r/\sqrt{d}\right),$$

where the ξ_j are i.i.d. with the one-dimensional marginal of $\hat{\pi}$ as its distribution. Then,

$$P\left(\left|\sum\nolimits_{j=1}^{n} \xi_j\right| \geq r/\sqrt{d}\right) \leq 2\exp\left[-nI\left(\frac{r}{\sqrt{d}n}\right)\right],$$

where

$$I(t) = \sup\left\{\lambda t - \log E\left(\mathrm{e}^{\lambda t}\right)\right\}.$$

By symmetry $I'(0) = 0$, and from our assumptions, we have $I''(0) \geq 1/Cm^2$. Furthermore, $I(t) = \infty$ if $|t| > 2$. By convexity of I, we therefore have $I(t) \geq t^2/Cm^2$. Combining these estimates gives

$$\sum_{x:|x|\geq r} \hat{\pi}_m^{*n}(x) \leq C \exp\left[\frac{r^2}{Cnm^2}\right].$$

From this, we get

$$\begin{aligned} \hat{\pi}^{*n}(x) &= \sum_y \hat{\pi}_m^{*(n-1)}(y)\, \hat{\pi}_m(x-y) \\ &\leq C m^{-d} \sum_{y:|y|\geq|x|-2m} \hat{\pi}_m^{*(n-1)}(y) \leq C m^{-d} \exp\left[-\frac{(|x|-2m)^2}{C(n-1)m^2}\right] \\ &\leq C m^{-d} \exp\left[-\frac{|x|^2}{Cnm^2}\right]. \end{aligned}$$

■

Let

$$G_m(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \hat{\pi}_m^{*n}(x). \tag{B.8}$$

Corollary B.3
For $|x| \geq m$, we have for some constant $c(d)$

$$G_m(x) = c(d) \frac{1}{\alpha(m)} |x|^{-d+2} + O\left(|x|^{-d} \left(\log \frac{|x|}{m}\right)^{5d}\right).$$

For $|x| \leq m$, we have

$$G_m(x) = \delta_{0,x} + O\left(m^{-d}\right).$$

Proof of Corollary B.3. Assume $|x| \geq m$ and set

$$N(x,m) \stackrel{\text{def}}{=} \frac{|x|^2}{\alpha(m)} \left(\log \frac{|x|^2}{\alpha(m)}\right)^{-10}.$$

Then

$$\sum_{n=N}^{\infty} \hat{\pi}^{*n}(x) = \sum_{n=N}^{\infty} \frac{1}{(2\pi d \alpha(m) n)^{d/2}} \exp\left[-\frac{|x|^2}{2\alpha(m) n}\right] + \sum_{n=N}^{\infty} O\left(\alpha(m)^{-d/2} n^{-(d+2)/2}\right).$$

and we note that

$$\sum_{n=N}^{\infty} O\left(\alpha(m)^{-d/2} n^{-(d+2)/2}\right) = O\left(|x|^{-d} \left(\log \frac{|x|^2}{\alpha(m)}\right)^{5d}\right).$$

Putting$t_n \stackrel{\text{def}}{=} 2\alpha(m) n / |x|^2$, we get

$$\begin{aligned}
&\sum_{n=N}^{\infty} \frac{1}{(2\pi d \alpha(m) n)^{d/2}} \exp\left[-\frac{|x|^2}{2\alpha(m) n}\right] \\
&= \frac{|x|^{-d+2}}{2(\pi d)^{d/2} \alpha(m)} \sum_{n=N}^{\infty} \frac{1}{(t_n)^{d/2}} \exp\left[-\frac{1}{t_n}\right] (t_n - t_{n-1}) \\
&= \frac{|x|^{-d+2}}{2(\pi d)^{d/2} \alpha(m)} \int_0^{\infty} t^{-d/2} \exp\left[-t^{-1}\right] dt + O\left(|x|^{-d}\right).
\end{aligned}$$

This proves the corollary for $|x| \geq m$ with

$$c(d) = \frac{1}{2(\pi d)^{d/2}} \int_0^{\infty} t^{-d/2} \exp\left[-t^{-1}\right] dt.$$

For $|x| \leq m$, the estimate is evident from Proposition B.1. ■

Proof of Lemma 3.9 a). There exists a θ, such that for any $y \in \mathrm{Shell}_L(r(L))$, there exists a unit vector $x \in \mathbb{R}^d$ such that $(y + C_\theta(x)) \cap \partial V_{3r(L)}(y) \cap V_L = \emptyset$. Using this, we see from (3.8), that our coarse grained Markov chain has after every visit of $\mathrm{Shell}_L(r(L))$ a probability of at least $\delta(\theta)$ to leave V_L in the next step. Therefore, the expected number of visits in this shell is finite, uniformly in the starting point.
■

Proof of Lemma 3.9 b). If $x \in \mathrm{Shell}_L(r, 2s)$, then $\hat{\pi}(x, \cdot)$ is an averaging over exit distributions from (discrete) balls $V_u(x)$, the averaging taken over u's with $u \geq (\gamma/2)\, d_L(x)$. Therefore, there exists a $\delta > 0$, such that $\hat{\pi}(x, \mathrm{Shell}_L(d_L(x)(1-\gamma/4))) \geq \delta$. Therefore, if $x \in \mathrm{Shell}_L(a, a+\gamma/8)$, $r(L) \leq a \leq 2s(L)$, we have $\hat{\pi}(x, \mathrm{Shell}_L(a(1-\gamma/8))) \geq \delta$. Therefore, a Markov chain with transition probabilities $\hat{\pi}$ which starts in $\mathrm{Shell}_L(a, a+\gamma s(L)/8)$ has probability at least δ to reach in one step $\mathrm{Shell}_L(a(1-\gamma/8))$. By Lemma 3.3 c), an nearest neighbor chain starting in $\mathrm{Shell}_L(a(1-\gamma/8))$ has a probability at least $\varepsilon(\gamma) > 0$ of exiting V_L before reentering into $\mathrm{Shell}_L(a, a+\gamma/8)$. This evidently then applies also to our coarse grained random walk.

We conclude that for the coarse grained chain starting in $x \in \mathrm{Shell}_L(a, a+\gamma s/8)$, there is a positive probability $\varepsilon > 0$, not depending on x, a, that the chain exits from V_L before reentering this shell. It therefore follows that the expected number of visits in $\mathrm{Shell}_L(a, a+\gamma s/8)$ is bounded, uniformly in the starting point of the chain, and a. From this the conclusion follows by summing over a finite number of such shells.
■

As a preparation for the proof of parts c) and d) of Lemma 3.9, we prove a preliminary result about our coarse grained random walk.

Lemma B.4

$$\sup_{x \in \mathrm{Shell}_L(2s(L))} \sum_{y \in V_{L-2s(L)}} \hat{g}_L(x, y) \leq C (\log L)^3 .$$

Proof of Lemma B.4. The expression $\sum_{y \in V_{L-2s(L)}} \hat{g}_L(x, y)$ is the expected total time that the random walk spends in $V_{L-2s} \subset V_L$. When starting in $\mathrm{Shell}_L(2s(L))$, the walk has a probability bounded from below, say by $\varepsilon_1 > 0$, of never entering $V_{L-2s(L)}$ before exiting V_L, uniformly in the starting point. If the walk enters $V_{L-2s(L)}$, it has to enter through $\mathrm{Shell}_L(2s, 4s)$. Therefore

$$\sup_{x \in \mathrm{Shell}_L(2s(L))} \sum_{y \in V_{L-2s}} \hat{g}_L(x, y) \leq \varepsilon_1^{-1} \left(1 + \sup_{x \in \mathrm{Shell}_L(2s(L), 4s(L))} E_x \left(T^{\mathrm{CG}}_{\mathrm{Shell}_L(2s(L))} \right) \right),$$

where T_A^{CG} stands for the first entrance time into A by the coarse grained random walk with transition kernel $\hat{\pi}_s$ from $V_{L-2s(L)}$. It therefore suffices to prove

$$\sup_{x \in \mathrm{Shell}_L(2s(L), 4s(L))} E_x \left(T^{\mathrm{CG}}_{\mathrm{Shell}_L(2s(L))} \right) \leq C (\log L)^3 ,$$

Consider the shells $R_j \overset{\mathrm{def}}{=} \mathrm{Shell}_L(js(L), (j+1)\, s(L))$, $j \geq 2$, and let T_j be the first entrance time of our (coarse grained) random walk into R_j.One then has

$$P_x \left(T^{\mathrm{CG}}_{R_j} < T^{\mathrm{CG}}_{\mathrm{Shell}_L(2s(L))} \right) \leq C P_x \left(T^{\mathrm{RW}}_{R_j} < T^{\mathrm{RW}}_{\mathrm{Shell}_L(2s(L))} \right),$$

and the right hand side we can estimate by Lemma 3.3 c), giving

$$P_x\left(T_{R_j}^{\mathrm{RW}} < T_{\mathrm{Shell}_L(2s(L))}^{\mathrm{RW}}\right) \leq \frac{C}{j},$$

and therefore we get

$$P_x\left(T_{R_j}^{\mathrm{CG}} < T_{\mathrm{Shell}_L(2s(L))}^{\mathrm{CG}}\right) \leq \frac{C}{j}.$$

If $x \in R_j$, we estimate the expected number of visits in R_j by Corollary B.3, which gives

$$\sup_{x\in R_j} \sum_{y\in R_j} G_{\gamma s(L)}(x,y) \leq C(\log L)^3.$$

Combining these estimates completes the proof of Lemma B.4 ■

Let σ be the first entrance time of $\{X_n\}$ into $\mathrm{Shell}_L(2s(L))$. Before time σ, the Markov process $\{X_n\}$ proceeds as a random walk on $\mathbb{Z}^d$ with jump distribution $\hat{\pi}_m$, where $m = \gamma s(L)$.

Proof of Lemma 3.9 c), d), e). From Corollary B.3, we get

$$\sup_{x\in V_L} \sum_{y\in V_{L-2s(L)}} G_{\gamma s(L)}(x,y) \leq C(\log L)^6.$$

Evidently, from Lemma B.4, we get

$$\sup_{x\in V_L} \sum_{y\in V_{L-2s(L)}} \left|G_{\gamma s(L)}(x,y) - \hat{g}_L(x,y)\right| \leq C(\log L)^3,$$

which implies the statement d).

e) follows by the same approximation and

$$\sup_{x,x'\in V:|x-x'|\leq s} \sum_{y\in \mathrm{Bulk}_L} \left|G_{\gamma s(L)}(x,y) - G_{\gamma s(L)}(x',y)\right| \leq C(\log L)^3,$$

which follows again from Corollary B.3.

We turn to the proof of part c). For $x = y$, the result is obvious from the transience of simple random walk. In the sequel, we thus always take $x \neq y$. Write $A_y \overset{\mathrm{def}}{=} \{z : |z-y| \leq s(L)\}$. We first prove the result for $x \in A_y$ and $d_L(y) \geq 5s(L)$. In that case,

$$\sup_{x\in A_y:x\neq y} \hat{g}_L(x,y) \leq G_{\gamma s(L)}(x,y) + \max_{z\in \mathrm{Shell}_L(2s(L))} P_z^{RW}(T_{A_y} < T_{V_L}) \sup_{x\in A_y:x\neq y} \hat{g}_L(x,y).$$

Since

$$\max_{z\in \mathrm{Shell}_L(2s(L))} P_z^{RW}(T_{A_y} < T_{V_L}) < 1$$

uniformly in L by Donsker's invariance principle, we conclude that

$$\sup_{x\in A_y:x\neq y} \hat{g}_L(x,y) \leq CG_{\gamma s(L)}(x,y).$$

Corollary B.3 then completes the proof in this case.

Consider next $x \in A_y$ but $s(L) \le d_L(y) \le 5s(L)$, and set $B_y \overset{\text{def}}{=} \{z : |z-y| \le s(L)/2\}$ and $C_y \overset{\text{def}}{=} \{z : |z-y| \le 5s(L)\}$. We note that

$$\sup_{x \in A_y : x \neq y} \hat{g}_L(x,y) \le \frac{C}{s(L)^d} + \sup_{x \notin A_y} \hat{g}_L(x,y) \le \frac{C}{s(L)^d} + \sup_{z \notin A_y} P_z^{\mathrm{RW}}(T_{B_y} < T_{V_L}) \sup_{x \in A_y : x \neq y} \hat{g}_L(x,y)\,.$$

Since $\sup_{z \notin A_y} P_z^{\mathrm{RW}}(T_{B_y} < T_{V_L}) < 1$ uniformly in L, again by Donsker's invariance principle, we conclude that

$$\sup_{x \in A_y : x \neq y} \hat{g}_L(x,y) \le \frac{C}{s(L)^d}\,,$$

which proves the claim in this case.

We next consider $x \notin A_y$. Let σ' denote the first entrance time of the simple random walk into $\mathrm{Shell}_L(2s(L))$. Clearly, $\sigma' \le \sigma$. We then have

$$\begin{aligned}
\hat{g}_L(x,y) &\le G_{\gamma s(L)}(x,y) + C \sum_{z \in \mathrm{Shell}_L(2s(L))} P_x^{\mathrm{RW}}(X_{\sigma'} = z) P_x^{\mathrm{RW}}(T_{A_y} < T_{V_L}) \sup_{w \in A_y : w \neq y} \hat{g}_L(w,y) \\
&\le \frac{C}{s(L)^2 |x-y|^{d-2}} + \frac{C d_L(x) d_L(y)}{s(L)^2} \sum_{z \in \mathrm{Shell}_L(2s(L))} \frac{1}{(|x-z| \vee 1)^d (|y-z| \vee 1)^d} \\
&\le \frac{C}{s(L)^2 |x-y|^{d-2}} + \frac{C}{s(L)^2} \sum_{z \in \mathrm{Shell}_L(2s(L))} \frac{1}{(|x-z| \vee 1)^{d-1} (|y-z| \vee 1)^{d-1}} \\
&\le \frac{C}{s(L)^2 |x-y|^{d-2}}\,,
\end{aligned} \tag{B.9}$$

where the second inequality uses Corollary B.3, the estimate on $\hat{g}_L(x,y)$ for $x \in A_y$ that was already proved, and Lemma 3.4. This completes the proof. ■

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